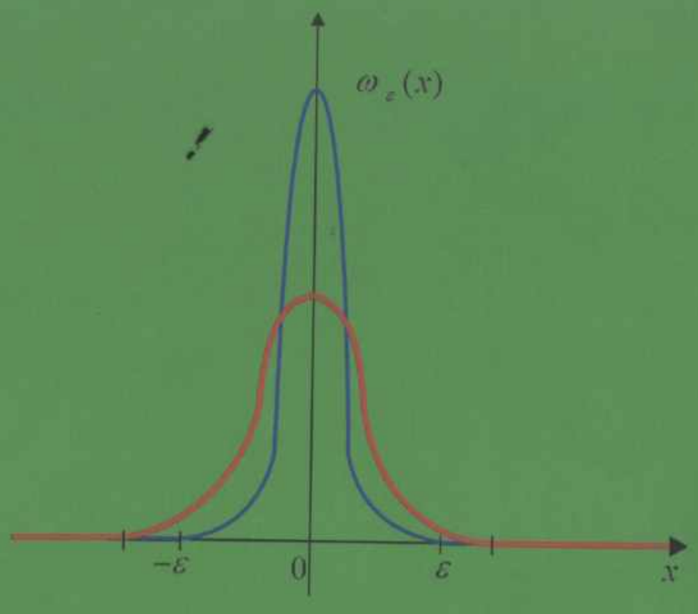
SH.G. KASIMOV

MATEMATIK FIZIKA TENGLAMALARI

I TOM

Xususiy hosilali differensial tenglamalarning sinflari. Matematik fizika tenglamalari uchun asosiy masalalarning qo'yilishi.



O'ZBEKISTON RESPUBLIKASI
OLIY TA'LIM, FAN VA INNOVATSIYA VAZIRLIGI

MIRZO ULUG'BEK NOMIDAGI
O'ZBEKISTON MILLIY UNIVERSITETI

SH.G.KASIMOV

MATEMATIK FIZIKA TENGLAMALARI

I TOM

O'QUV QO'LLANMA

60540100 – Matematika va 60540200 – Amaliy matematika ta'lim yo'nalishi talabalari uchun

Toshkent
"Ma'rifat"
2024

UO‘K: 517.958(075.8)
KBK: 22.311ya73
K 27

**Kasimov Sh.G. “Matematik fizika tenglamalari. I Tom”.
O‘quv qo‘llanma. –T.: “Ma’rifat”, 2024. 912 bet.**

O‘quv qo‘llanmada xususiy hosilali tenglamalarning xarakteristikasi haqida tushuncha, xususiy hosilali tenglamalarning klassifikatsiyasi va ularning kanonik ko‘rinishi, matematik fizikaning asosiy tenglamalari, asosiy chegaraviy masalalarning qo‘yilishi, Koshi masalasi, matematik fizika masalalarining korrekt qo‘yilishi kabi mavzularga oid ma’ruzalar va mashqlar bayon qilingan.

Ushbu o‘quv qo‘llanma universitetlarning mexanika-matematika, amaliy matematika va intellektual texnologiyalar fakultetlarining bakalavr ta’lim yo‘nalishlari talabalari uchun mo‘ljallangan bo‘lib, undan “Matematika” mutaxassisligi bo‘yicha magistrlar tayyorlaydigan fakultetlarning magistrantlari, mustaqil izlanuvchilar va doktorantlari ham foydalanishlari mumkin.

UO‘K: 517.958(075.8)
KBK: 22.311ya73
K 27

Taqrizchilar:
Mamatov M.Sh. – fizika-matematika fanlari doktori, professor
Djamalov S.Z. – fizika-matematika fanlari doktori, dotsent
Pirmatov Sh.T. – fizika-matematika fanlari nomzodi, dotsent

O‘quv qo‘llanma Mirzo Ulug‘bek nomidagi O‘zMU o‘quv-uslubiy Kengashida 2024 yil 31 maydagi № 8-sonli bayonnomasi bilan muhokama qilingan va chop etishga tavsiya qilingan.

ISBN: 978-9910-765-82-7

Ushbu o'quv qo'llanmani bobom Yusupov Allanazar va buvim Xo'jaqova Jumagullarning porloq xotiralariga bag'ishlayman.

So'z boshi

Mazkur o'quv qo'llanma universitetlarning mexanika-matematika, amaliy matematika va intellektual texnologiyalar fakultetlarining bakalavr yo'nalishlari uchun mo'ljallangan "matematik fizika tenglamalari" kursi dasturiga moslab yozilgan.

Ushbu o'quv qo'llanmani yozishda mashhur rus va boshqa chet el olimlari tomonidan yaratilgan darslik va monografiyalardan ham keng foydalanildi. Shuningdek, muallif o'quv qo'llanmani yozishda Urganch Davlat universitetining fizika-matematika fakultetida, Mirzo Ulug'bek nomidagi O'zbekiston Milliy unuversitetining mexanika-matematika, amaliy matematika va intellektual texnologiyalar fakultetlarida, M.V. Lomonosov nomidagi Moskva Davlat universitetining Toshkent filialida, hamda "Moskva muhandislik-fizika instituti" Milliy tadqiqot yadro universitetining Toshkent filialida "matematik fizika tenglamalari" kursi bo'yicha o'qigan ma'ruzalari va olib borgan amaliyot darslarida qo'llagan mashqlaridan ham foydalandi.

Matematik fizika masalalarining doirasi nihoyatda keng bo'lib, ular turli fizik, mexanik, texnik, biologik va boshqa jarayonlarni o'rganish bilan uzviy bog'liqdir. Ushbu o'quv qo'llanmada matematik fizikaning xususiy hosilali differensial tenglamalarga keladigan masalalari tekshiriladi. Asosiy e'tibor matematik fizika tenglamalarining uchta klassik: elliptik, giperbolik va parabolik tipdagi tenglamalarini o'rganishga qaratilgan. Bu tenglamalarni tekshirish integral tenglamalar nazariyasi, umumlashgan funksiyalar nazariyasi, teskari va nokorrekt qo'yilgan masalalar nazariyasi, Gilbert va Banax fazolaridagi chiziqli operator tenglamalar nazariyasi, nochiziqli operator tenglamalar nazariyasi, aralash tipdagi tenglamalar nazariyasi, differensial operatorlarning spektral nazariyasi, matematik fizikaning variatsion usullari va maxsus funksiyalari kabi nazariyalar bilan uzviy bog'liq holda bayon qilingan.

Shuningdek, xususiy hosilali differensial tenglamalarni yechishda tez-tez qo'llaniladigan bir qator usullar, jumladan, o'zgaruvchilarni ajratish, untegral almashtirishlar usullari, potentsiallar usuli, variatsion usullarni qo'llab yechiladigan masalalar ham keltirilgan.

Ushbu o'quv qo'llanma universitetlarning mexanika-matematika, amaliy matematika va intellektual texnologiyalar fakultetlarining bakalavr yo'nalishlari talabalari uchun mo'ljallangan bo'lib, undan "Matematika" mutaxassisligi bo'yicha magistrlar tayyorlaydigan fakultetlarning magistrantlari, mustaqil izlanuvchilar va doktorantlari ham foydalanishlari mumkin.

Muallif

Kirish

Matematik fizika – bu fizik hodisalarning matematik modellari nazariyasidir. Bu fan matematikaga tegishli bo'lib uning haqiqatlik kriteriyasi – bu matematik isbotdir. Biroq, sof matematik fanlardan farqi shundan iboratki, matematik fizika fani fizik masalalarni matematika darajasida tadqiq etadi va natijalar teoremalar, grafiklar, jadvallar va hokozo shakllarda ifodalanadi, hamda fizik tasavvurlar hosil qilinadi. Matematik fizikaning bunday keng ma'noda tushunilishi unga mexanikaning nazariy mexanika, gidrodinamika va elastiklik nazariyasi kabi bo'limlarining ham ta'luqliligini bildiradi.

Dastlabki matematik fizika masalalari differensial (integro–differensial) tenglamalar uchun chegaraviy masalalarni yechishga olib kelingan. Bu yo'nalish *klassik matematik fizika* predmetini tashkil etadi va hozirda ham o'zining muhim ahamiyatini saqlab turibdi.

Klassik matematik fizika I. Nyuton davridan boshlab fizika va matematikaning parallel rivojlanishi bilan birga rivojlanib bordi. XVII asrning oxirlarida differensial va integral hisob (I. Nyuton, G. Leybnits) yaratildi va klassik mexanikaning asosiy qonunlari, hamda butun olam tortishish qonuni (I. Nyuton) ifoda qilindi. XVIII asrda tor, sterjen, mayatniklarning tebranishi, hamda akustika va gidrodinamika bilan bog'liq masalalarni o'rganish uchun matematik fizikaning usullari shakllana boshladi. Shuningdek analitik mexanikaning asoslari (J. Dalamber, L. Eyler, D. Bernulli, J. Lagranj, K. Gauss, P. Laplas) yaratildi. XIX asrda matematik fizika usullari issiqlik o'tkazuvchanlik, diffuziya, elastiklik nazariyasi, optika, elektrodinamika, nochiziqli to'lqin jarayonlari va hokozo masalalar bilan bog'liq bo'lgan yangi rivojlanishiga erishdi. Potensiallar nazariyasi, harakatning turg'unlik nazariyasi (J. Furye, S. Puasson, L. Bolsman, O. Koshi, M.V. Ostrogradskiy, P. Dirixle, Dj.K. Maksvell, B. Riman, S.V. Kovalevskaya, D. Stoks, G.R. Kirxgof, A. Puankare, A.M. Lyapunov, V.A. Steklov, D. Gilbert, J. Adamar) yaratildi. XX asrga kelib kvant fizikasi va nisbiylik nazariyasining masalalari, hamda gaz dinamikasi, zarrachalarning ko'chish nazariyasi va plazma fizikasining yangi muammolari ham matematik fizikaga kirib keldi.

Klassik matematik fizikada uch xil tipdagi sodda differensial tenglamalar Puasson (xususan Laplas) tenglamasi, issiqlik o'tkazuvchanlik tenglamasi, to'lqin tenglamasi bilan bog'liq bo'lgan har xil masalalar o'rganilgan.

Kvant mexanikasi va yadroviy energetikaning rivojlanishi bilan matematik fizikaning yangi tipdagi tenglamalari va chegaraviy masalalari paydo bo'ldi. Bular qatoriga to'lqin funksiyasi uchun Shryodinger tenglamasi, statsionar Shryodinger tenglamasi, Gelmgols tenglamasi va bu tenglama uchun Zommerfeld nurlanish shartlarini qanoatlantiruvchi masala, izotrop sochilish uchun zarrachalar ko'chishining bir xil tezlikli tenglamasini keltirish mumkin.

Bu masalalarni tadqiq etishda oddiy va xususiy hosilali differensial tenglamalar nazariyasi, integral tenglamalar, variatsion hisob, funksiyalar nazariyasi, funksional analiz, ehtimollar nazariyasi, taqribiy usullar va hisoblash matematikasi asosiy matematik qurol bo'lib xizmat qiladi.

XX asrda kvant fizikasining yangi bo'limlari: kvant mexanikasi, kvant maydon nazariyasi, kvant statistik fizikasi, nisbiylik nazariyasi, gravitatsiya kabi (A. Puankare, D. Gilbert, P. Dirak, A. Eynshteyn, N.N. Bogolyubov, V.A. Fok, E. Shryodinger, G. Veyl, R. Feynman, Dj. fon Neyman, V. Geyzenberg) bo'limlar paydo bo'ldi. Bu hodisalarni o'rganish uchun qo'llaniladigan matematik qurollar to'plami sezilarli ravishda kengaydi. Matematikaning an'anaviy sohalari bilan bir qatorda operatorlar nazariyasi, umumlashgan funksiyalar nazariyasi, ko'p kompleks o'zgaruvchili funksiyalar nazariyasi, topologik va algebraik usullar, sonlar nazariyasi, p–adik analiz, asimptotik va hisoblash usullari keng qo'llanila boshlandi. Elektron hisoblash mashinalarining paydo bo'lishi bilan batafsil tahlil qilinadigan matematik modellarning juda muhim sinflari kengayib bordi. Hisoblash eksperimentini o'tkazishning real imkoniyatlari paydo bo'ldi. Masalan, atom bombasining portlashini modellashtirish yoki atom reaktorining real vaqt masshtabidagi ishini modellashtirish mumkin bo'ldi. Zamonaviy nazariy fizika va zamonaviy matematikaning bunday intensiv o'zaro ta'sirida yangi soha – *zamonaviy matematik fizika* shakllandi. Uning modellari hamma vaqt ham differensial tenglamalar uchun qo'yilgan chegaraviy masalalarga keltirilavermaydi, balki aksiomalar sistemasi shaklida ifoda qilinadi. Matematikada, ayniqsa geometriyada va to'plamlar nazariyasida aksiomatik usul ancha oldindan ma'lum edi. Har qanday aksiomalar sistemasi singari, bunday sistema qarama–qarshiliksiz, bog'liqmaslik, quriluvchanlik va to'lalik talablarini qanoatlantirishi kerak bo'ladi.

XX asrga kelib nazariy fizikaning rivojida bu tendensiyani P. Dirak yaxshi tushundi. U 1930 yildagi o'z maqolasida nazariy jihatdan pozitronning mavjudligini aytdi. U quyidagicha yozadi: "Ehtimol mening fikrimcha bunday uzluksiz abstraktlash jarayoni davom etib boradi va kelajakda fizikaning yutug'i ko'p darajada uzluksiz modifikatsiyalashga va matematika asosida aksiomalarni umumlashtirishga asoslangan bo'ladi". Nazariy fizikaning keyingi taraqiyoti P. Dirakning fikrlarini to'la tasdiqladi. Bunga yorqin misol sifatida nazariy fizikada aksiomalashtirish usullarining qo'llanilishi N.N. Bogolyubov tomonidan o'tgan asrning 50–yillarida kvant maydon nazariyasida aksiomalashtirishda sezildi. Shu davrda Gamilton formalizmini qo'llaganda ultrafiolet uzoqlashuv muammosi bor edi. Bu muammoga N.N. Bogolyubov boshqacha yondashuv bilan qarashni taklif etdi. U avval Gamilton formalizmidan voz kechdi va Geyzenberg tomonidan kiritilgan sochilishning matritsaviy nazariyasini asos qilib qabul qildi. N.N. Bogolyubov bu bilan mumkin bo'lgan matematik obyektlar to'plamini kengaytirdi. Bunda sochilish matritsasining elementlari deb operator qiymatli umumlashgan funksiyalar olindi. Shu bilan birga sochilish matritsasi *relyativistlik, kovariantlik, unitarlik, sabablilik, spektrallik* kabi asosiy fizik postulatlarni qanoatlantirishi talab qilindi.

Matematik fizikaning masalalari orasida J. Adamar bo'yicha *korrekt qo'yilgan masalalar*, ya'ni yechimi mavjud, yagona va shu masaladagi berilganlarga uzluksiz bog'liq bo'lgan masalalar juda muhim sinflardan biri sifatida ajratiladi. Bu talablar birinchi qarashda juda tabiiydek bo'lib ko'rinsada, ularni qabul qilingan matematik modellar darajasida isbot qilish zarur bo'ladi. Masala korrektligining isboti – bu birinchidan shu matematik modelning qo'llanilishini bildiradi: model qarama–qarshiliksiz (yechim mavjud), model bir qiymatli ravishda fizik jarayonni ifoda qiladi (yechim yagona), model fizik miqdorlarning chetlashuvida kichik sezgirlikka ega (yechim masalada berilganlarga uzluksiz bog'liq bo'ladi).

Matematik fizika masalalarini tadqiq etishda umumlashgan funksiyalar muhim rol o'ynaydi va bu umumlashgan yechim bilan jips bog'langandir. Shu sababli umumlashgan funksiyalar nazariyasi muhim ahamiyatga egadir.

Mazkur o'quv qo'llanmaning 1-qismida xususiy hosilali differensial tenglamalarning xarakteristikasi haqida tushuncha, xususiy hosilali differensial tenglamalarning klassifikatsiyasi va ularning kanonik ko'rinishi, ikkinchi tartibli chiziqli differensial tenglamalar uchun asosiy chegaraviy masalalarning qo'yilishi, Koshi masalasi yechimining mavjudligi va yagonaligi haqidagi S.V. Kovalevskaya teoremasi, matematik fizika masalalarining korrekt qo'yilishi, korrekt va nokorrekt masalalar, A.N. Tixonovning regulyarlashtirish metodi, xarakteristik ko'phad ildizlari terminida Koshi masalasining nokorrekt qo'yilish shartlari batafsil bayon qilingan.

Har bir bob paragraflarga bo'lib chiqilgan. Paragraflar punktlarga bo'lib chiqilgan va har birida mavzuga oid asosiy tushunchalar keltirilgan, tegishli teoremalar isbotlari bilan berilgan va unga doir namunaviy misollar tahlil qilingan. Paragraflar mavzuga oid mustaqil ish uchun vazifalar bilan boyitilgan.

I BOB
XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALARNING SINFLARI. MATEMATIK FIZIKA TENGLAMALARI UCHUN ASOSIY MASALALARNING QO'YILISHI

Matematik fizikaning ko'pgina masalalari xususiy hosilali differensial tenglamalarga olib kelinadi. Eng ko'p qo'llaniladigan differensial tenglamalar – bu ikkinchi tartibli differensial tenglamalardir. Ushbu bobda bunday chiziqli differensial tenglamalarning klassifikatsiyasi va ularning kanonik ko'rinishi, ikkinchi tartibli chiziqli differensial tenglamalar uchun asosiy chegaraviy masalalarning qo'yilishi, Koshi masalasi, matematik fizika masalalarining korrekt qo'yilishi, korrekt va nokorrekt masalalar kabi mavzularni ko'rib chiqamiz.

1 - §. Xususiy hosilali differensial tenglamalarning xarakteristikasi haqida tushuncha. Xususiy hosilali differensial tenglamalarning klassifikatsiyasi va ularning kanonik ko'rinishi

1. Xususiy hosilali differensial tenglama tushunchasi va uning yechimi. Ω – orqali $x_1, x_2, ..., x_n$, $n \geq 2$, ortogonal dekart koordinatali x nuqtalarning $n-$ o'lchamli R^n evklid fazosidan olingan sohani belgilaymiz.

Ω – sohadan olingan x nuqta va $\alpha_1, \alpha_2, ..., \alpha_n$, $\sum_{j=1}^{n} \alpha_j = k$, $k = 0,...,m$, $m \geq 1$, manfiymas butun indeksli $p_{\alpha_1 \alpha_2 ... \alpha_n}$ haqiqiy o'zgaruvchili $F(x,..., p_{\alpha_1 \alpha_2 ... \alpha_n},...)$ – haqiqiy qiymatli funksiya berilgan bo'lib, hech bo'lmaganda $\dfrac{\partial F}{\partial p_{\alpha_1 \alpha_2 ... \alpha_n}}$, bunda $\sum_{j=1}^{n} \alpha_j = m$, hosilalardan birortasi noldan farqli bo'lsin. Bu yerda $p_{\alpha_1\alpha_2...\alpha_n} = \dfrac{\partial^m u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}$ deb olamiz.

$$F\left(x,...,\frac{\partial^k u}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}},...\right)=0$$

shakldagi tenglik $u(x)=u(x_1,x_2,...,x_n)$, $x\in\Omega$ noma'lum funksiy nisbatan $m-$tartibli xususiy hosilali differensial tenglama deyiladi va tenglikning chap tomoni esa, $m-$tartibli xususiy hosilali differensia operator deyiladi.

Ω sohada aniqlangan $u(x)$ haqiqiy qiymatli funksiya va uning (1) tenglamada qatnashgan barcha xususiy hosilalari uzluksiz bo'lib, bu tenglamani ayniyatga aylantirsa, u holda shu funksiyaga regulyar yechim deyiladi.

Agar F funksiya $p_{\alpha_1\alpha_2...\alpha_n}$, bunda $|\alpha|=\sum_{j=1}^{n}\alpha_j=k$, $k=0,...,m$ barcha o'zgaruvchilarga nisbatan chiziqli funksiya bo'lsa, u holda (1) tenglamaga chiziqli tenglama deb ataladi. Agar F funksiya $p_{\alpha_1\alpha_2...\alpha_n}$, bunda $|\alpha|=\sum_{j=1}^{n}\alpha_j=m$ o'zgaruvchigagina nisbatan chiziqli bo'lsa, u holda (1) tenglamaga kvazichiziqli tenglama deb ataladi.

$Lu=f(x)$ chiziqli tenglama uning o'ng tomonidagi $f(x)$ funksiyaning barcha $x\in\Omega$ uchun nolga teng yoki aynan noldan farqli bo'lishligiga qarab bir jinsli yoki bir jinsli bo'lmagan tenglama deb ataladi.

Osongina ko'rsatish mumkinki, agar $u(x)$ va $v(x)$ funksiyalar bir jinsli bo'lmagan $Lu=f(x)$ chiziqli tenglamaning yechimlari bo'lsa, u holda ularning ayirmasi $w(x)=u(x)-v(x)$ esa $Lw=0$ bir jinsli tenglamaning yechimi bo'ladi. Bundan tashqari, agar $u_k(x)$, $k=1,...,l$ funksiyalar bir jinsli tenglamaning yechimlari bo'lsa, u holda

$u=\sum_{k=1}^{l} c_k u_k(x)$, bunda c_k – haqiqiy o'zgarmaslar, ham shu tenglamaning yechimi bo'ladi.

2. Xususiy hosilali differensial tenglamalarning xarakteristikasi haqida tushuncha. (1) tenglamaning tipi

$$K(\xi_1,\dots,\xi_n)=\sum_{|\alpha|=m}\frac{\partial F}{\partial p_{\alpha_1\dots\alpha_n}}\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}$$

xarakteristik forma orqali aniqlanadi.

$$\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha u=f(x) \qquad (2)$$

tenglama m – tartibli xususiy hosilali chiziqli differensial tenglamaning umumiy shakli, bunda $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ – multiindeks, $\alpha_j\ge 0$, $j=\overline{1,\ n}$, bundan tashqari α_j – butun sonlar, $|\alpha|=\alpha_1+\alpha_2+\dots+\alpha_n$ multiindeks moduli, $D^\alpha u=\dfrac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$ bo'lsin. $D^\alpha\to\xi^\alpha$ almashtirish orqali (2) tenglamaning $\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ simvolini hosil qilamiz, bunda $\xi^\alpha=\xi_1^{\alpha_1}\xi_2^{\alpha_2}\dots\xi_n^{\alpha_n}$. $\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ formaga (2) tenglamaning bosh simvoli yoki xarakteristik ko'phadi deb ataladi.

$x\in\Omega$ tayinlangan nuqta bo'lsin. Noldan farqli $\xi=(\xi_1,\xi_2,\dots,\xi_n)\ne 0$ vektor uchun $\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha=0$ bo'lsa, u holda bu vektor xarakteristik yo'nalish deb ataladi.

$F(x_1,x_2,\dots,x_n)=0$ formula bilan berilgan gipersirt uchun har bir nuqta xarakteristik yo'nalishga ega bo'lsa, ya'ni

$$\begin{cases} F(x_1,x_2,\dots,x_n)=0 \\ \sum_{|\alpha|=m} a_\alpha(x)\left(\dfrac{\partial F}{\partial x_1}\right)^{\alpha_1}\dots\left(\dfrac{\partial F}{\partial x_n}\right)^{\alpha_n}=0,\ \ gradF\ne 0 \end{cases} \qquad (3)$$

bo'lsa, u holda xarakteristik sirt deb ataladi. Bu xarakteristika tenglamasidir, bunda $grad\,F = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, ..., \frac{\partial F}{\partial x_n}\right)$.

Ikkinchi tartibli kvazichiziqli (barcha yuqori tartibli hosilalarga nisbatan chiziqli) uzluksiz $a_{ij}(x)$ koeffitsientli tenglamani qaraymiz:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \Phi(x, u, grad\,u) = 0. \qquad (4)$$

$x = (x_1, x_2, ..., x_n)$, $n \geq 2$ o'zgaruvchili $F(x)$ funksiya C^1 sinfdan olingan bo'lib, $F(x) = 0$ sirtda $grad\,F(x) \neq 0$ va

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial F}{\partial x_i}\cdot\frac{\partial F}{\partial x_j} = 0 \qquad (5)$$

bo'lsin. U holda $F(x) = 0$ esa, (4) kvazichiziqli differensial tenglamaning xarakteristik sirti deb, (5) tenglama esa xarakteristik tenglamasi deb aytiladi. $n = 2$ uchun xarakteristik sirti xarakteristik chiziq deb ataladi.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Delta u$$

to'lqin tarqalish tenglamasi uchun xarakteristik tenglama

$$F = 0 \text{ da } \left(\frac{\partial F}{\partial t}\right)^2 - a^2\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2 = 0$$

shaklga egadir.

Uchi (x_0, t_0) nuqtada bo'lgan xarakteristik konus deb ataluvchi

$$a^2(t - t_0)^2 - |x - x_0|^2 = 0$$

sirt xarakteristik sirt bo'ladi.

$$\frac{\partial u}{\partial t} = a^2 \Delta u + f$$

issiqlik o'tkazuvchanlik tenglamasi uchun xarakteristik tenglama

$$F = 0 \text{ da } -a^2\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2 = 0$$

shaklga egadir. Uning xarakteristikalari osongina ko'rinadiki, $t = C$ tekisliklar oilasidan iborat bo'ladi.

$$\Delta u = f$$

Puasson tenglamasi uchun xarakteristik tenglama

$$F = 0 \quad \text{da} \quad \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \right)^2 = 0$$

shaklga egadir. Bundan $F = 0$ da $grad\, F = 0$ ekanligi kelib chiqadi, bu esa mumkin emas, ya'ni xarakteristik sirtga ega emas.

3. Misollar. Endi xususiy hosilali differensial tenglamalarning tartibini va uning chiziqli differensial tenglama ekanligini aniqlashga doir misollar keltiramiz:

1–misol. $\sin(u_x - u_y) - \sin u_x \cos u_y + \sin u_y \cos u_x = 0$ tenglama xususiy hosilali differensial tenglama bo'ladimi?

Yechish:

$$\frac{\partial F}{\partial u_x} = \frac{\partial\left(\sin(u_x - u_y) - \sin u_x \cos u_y + \sin u_y \cos u_x\right)}{\partial u_x} =$$

$$= \cos(u_x - u_y) - \cos u_x \cos u_y - \sin u_y \sin u_x =$$

$$= \cos u_x \cos u_y + \sin u_y \sin u_x - \cos u_x \cos u_y - \sin u_y \sin u_x = 0.$$

Xuddi shunday, $\frac{\partial F}{\partial u_y} = 0$ ekanligini ko'rsatish mumkin. F funksiyadan xususiy hosilalar bo'yicha olingan hosilalar nolga teng ekan. Demak, berilgan tenglama xususiy hosilali differensial tenglama bo'lmas ekan.

2–misol. $u_{xx}^2 + u_{yy}^2 - \left(u_{xx} - u_{yy}\right)^2 = 0$ tenglama xususiy hosilali differensial tenglama bo'ladimi?

Yechish: Avval berilgan tenglamani soddalashtirib olaylik.

$$F = u_{xx}^2 + u_{yy}^2 - \left(u_{xx} - u_{yy}\right)^2 = u_{xx}^2 - u_{yy}^2 - u_{xx}^2 +$$

$+2u_{xx}u_{yy} - u_{yy}^2 = 2u_{xx}u_{yy}$. Endi esa, xususiy hosilalar bo'yicha hosilalarni hisoblaymiz:

$$\frac{\partial F}{\partial u_{xx}} = 2u_{yy} \neq 0, \quad \frac{\partial F}{\partial u_{yy}} = 2u_{xx} \neq 0.$$

Demak, berilgan tenglama 2–tartibli xususiy hosilali differensial tenglama ekan.

3–misol. $\cos^2 u_{xx} + \sin^2 u_{xx} + 4u_x^3 - 2u_y + u = 0$ tenglamaning tartibini aniqlang.

Yechish:

$$\frac{\partial\left(\cos^2 u_{xx} + \sin^2 u_{xx} + 4u_x^3 - 2u_y + u\right)}{\partial u_{xx}} =$$

$$= -2\cos u_{xx} \sin u_{xx} + 2\sin u_{xx} \cos u_{xx} = 0$$

$$\frac{\partial\left(\cos^2 u_{xx} + \sin^2 u_{xx} + 4u_x^3 - 2u_y + u\right)}{\partial u_x} = 12u_x^2 \neq 0$$

Demak, berilgan tenglama 1–tartibli xususiy hosilali differensial tenglama ekan.

4–misol. $x^2 u_{xx} + xu_{xy} + \sin y \cdot u_x - y^2 u = 0$ tenglama xususiy hosilali chiziqli differensial tenglama bo‘ladimi?

Yechish:

$$\frac{\partial\left(x^2 u_{xx} + xu_{xy} + \sin y \cdot u_x - y^2 u\right)}{\partial u_{xx}} = x^2 \neq 0$$

bo‘lgani uchun berilgan tenglama 2–tartiblidir. Hamda tenglama $Lu = f$ ko‘rinishida bo‘lib, bu yerda

$$Lu = x^2 u_{xx} + xu_{xy} + \sin y \cdot u_x - y^2 u$$

operator noma’lum funksiya va uning hosilalari u_{xx}, u_{xy}, u_x, u larga nisbatan chiziqlidir, chunki

$$L(\alpha u + \beta v) = x^2(\alpha u + \beta v)_{xx} + x(\alpha u + \beta v)_{xy} +$$

$$+\sin y\cdot(\alpha u+\beta v)_x - y^2(\alpha u+\beta v) =$$

$$=\alpha x^2 u_{xx} + \alpha x u_{xy} + \alpha\sin y\cdot u_x - \alpha y^2 u + \beta x^2 v_{xx} +$$

$$+\beta v_{xy} + \beta\sin y\cdot v_x - \beta y^2 v = \alpha Lu + \beta Lv$$

va tenglamaning o'ng tomoni $f(x)=0$ dir. Shuning uchun 2–tartibli xususiy hosilali bir jinsli chiziqli differensial tenglama bo'ladi.

5–misol. $xu_{xx}^2 + (x+y)u_{xy} - 5y\cdot u_x - y^2 u + \sin(x+y) = 0$ tenglama xususiy hosilali chiziqli differensial tenglama bo'ladimi?

Yechish: Berilgan tenglama xususiy hosilali chiziqli differensial tenglama bo'la olmaydi, chunki $Lu = xu_{xx}^2 + (x+y)u_{xy} - 5y\cdot u_x - y^2 u$ operator u_{xx} ga nisbatan birinchi tartibli (chiziqli) emas.

6–misol. $xu_{xxy} + y^2 u_{yyy} - 5y\cdot u_{xy} - \ln y\cdot u_x - y^2 u +$ $+\sin(x+y) + x^2 = 0$ tenglama xususiy hosilali chiziqli differensial tenglama bo'ladimi?

Yechish: Berilgan tenglama xususiy hosilali bir jinsli bo'lmagan chiziqli differensial tenglama bo'ladi, chunki

$$Lu = xu_{xxy} + y^2 u_{yyy} - 5y\cdot u_{xy} - \ln y\cdot u_x - y^2 u$$

operator u_{xxy}, u_{yyy}, u_{xy}, u_x, u larga nisbatan birinchi darajali differensial ifoda va $f(x) = -\sin(x+y) - x^2 \neq 0$.

7–misol. $u_{xxy} - tgy\cdot u_{yy} + 7y\cdot u_{xy}u_{yy} - y\cdot u_x - yu + xy = 0$ tenglama xususiy hosilali chiziqli differensial tenglama bo'ladimi?

Yechish: Berilgan tenglama xususiy hosilali chiziqli differensial tenglama bo'la olmaydi, chunki

$$Lu = u_{xxy} - tgy\cdot u_{yy} + 7y\cdot u_{xy}u_{yy} - y\cdot u_x - yu$$

operator u_{xy}, u_{yy} larga nisbatan birinchi darajali differensial ifoda emas. Ammo ushbu tenglama xususiy hosilali kvazichiziqli differensial tenglama bo'ladi, chunki $Lu = u_{xxy} - tgy\cdot u_{yy} + 7y\cdot u_{xy}u_{yy} - y\cdot u_x - yu$ operator yuqori tartibli hosila u_{xxy} ga nisbatan birinchi darajali differensial ifoda bo'ladi.

4. Xususiy hosilali differensial tenglamalarning klassifikatsiyasi va ularning kanonik shakli.

Ta'rif. *Agar har qanday* $|\xi| \neq 0$ *uchun* $\sum_{|\alpha|=m} a_\alpha(x^0)\xi^\alpha \neq 0$

(boshqacha qilib aytganda, uning haqiqiy xarakteristikasi yo'q) bo'lsa, u holda

$$\sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u = f(x)$$

chiziqli tenglama x^0 *nuqtada elliptik tipdagi tenglama deyiladi.*

Ta'rif. *Agar* $\sum_{j=1}^{n-1} \xi_j^2 \neq 0$ *bo'ladigan har qanday* $\xi_1,\dots,\xi_{n-1}$ *uchun*

$\sum_{|\alpha|=m} a_\alpha(x^0)\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n} = 0$ *tenglama* ξ_n *o'zgaruvchiga nisbatan* m

ta haqiqiy va har xil ildizlarga ega bo'lsa, u holda

$$\sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u = f(x)$$

chiziqli tenglama x^0 *nuqtada* x_n *o'q yo'nalishida giperbolik tipdagi tenglama deyiladi.*

Ta'rif. *Agar* x^0 *tayinlangan nuqta uchun shunday bir*

$$\xi_i = \xi_i(\mu_1,\dots,\mu_n),\ i = 1,2,\dots,n,$$

o'zgaruvchilarning affin almashtirishini topish mumkin bo'lib, natijada, $\sum_{|\alpha|=m} a_\alpha(x^0)\,\xi^\alpha$ *forma* μ_i *o'zgaruvchilarning faqatgina* l *–tasinigina, bunda* $0 < l < n$, *saqlasa, u holda* $\sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u = f(x)$ *tenglama* x^0 *nuqtada parabolik maxsuslikka ega yoki parabolik tipdagi tenglama deyiladi.*

Ikkinchi tartibli xususiy hosilali chiziqli tenglamani quyidagi shaklda yozish mumkin:

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}(x)\cdot u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\cdot u_{x_i} + c(x)u + f(x) = 0,$$

$(a_{ij}=a_{ji})$, bunda $a,\ b,\ c,\ f$ funksiyalar $x=(x_1,x_2,...,x_n)$ o‘zgaruvchiga bog‘liqdir. Yangi ξ_k erkli o‘zgaruvchilarni $\xi_k=\xi_k(x_1,x_2,...,x_n),\ \ k=\overline{1,n}$ shaklda kiritamiz. U holda

$$u_{x_i}=\sum_{k=1}^{n}u_{\xi_k}\cdot\alpha_{ik},\quad u_{x_ix_j}=\sum_{k=1}^{n}\sum_{l=1}^{n}u_{\xi_k\xi_l}\cdot\alpha_{ik}\alpha_{jl}+\sum_{k=1}^{n}u_{\xi_k}\cdot\left(\xi_k\right)_{x_ix_j}$$

bunda $\alpha_{ik}=\dfrac{\partial\xi_k}{\partial x_i}$. Hosila uchun olingan ifodalarni berilgan tenglamaga qo‘ysak, quyidagini hosil qilamiz:

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\overline{a}_{kl}\cdot u_{\xi_k\xi_l}+\sum_{k=1}^{n}\overline{b}_k\cdot u_{\xi_k}+cu+f=0,$$

bunda

$$\overline{a}_{kl}=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\alpha_{ik}\alpha_{jl},\quad \overline{b}_k=\sum_{i=1}^{n}b_i\cdot\alpha_{ik}+\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\cdot\left(\xi_k\right)_{x_ix_j}.$$

Endi

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(x^0)y_iy_j$$

kvadratik formani qaraymiz. y o‘zgaruvchi ustida

$$y_i=\sum_{k=1}^{n}\alpha_{ik}\eta_k$$

chiziqli almashtirish bajarib,

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\overline{a}_{kl}(x^0)\,\eta_k\eta_l$$

kvadratik formaga ega bo‘lamiz, bunda

$$\overline{a}_{kl}(x^0)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(x^0)\,\alpha_{ik}\,\alpha_{jl}$$

bo‘ladi.

Ma’lumki, chiziqli almashtirishni mos tanlash yo‘li bilan kvadratik formaning $a_{ij}(x^0)$ matritsasini diagonal shaklga, ya’ni $\sum_{i=1}^{n}\alpha_i\cdot\eta_i^2$

kanonik shaklga keltirish mumkin bo'lib, bunda α_i, $i=\overline{1,n}$ koeffitsientlar 1, −1, 0 qiymatlarni qabul qiladi, bundan tashqari inertsiya qonuniga ko'ra, musbat, manfiy va nolga teng koeffitsientlar soni kvadratik formani kanonik shaklga keltirishdagi chiziqli almashtirishga nisbatan invariantdir.

Agar barcha n ta α_i koeffitsientlar bir xil ishorali bo'lsa, u holda tenglama x^0 nuqtada elliptik tipdagi tenglama deb, agar $n-1$ ta α_i koeffitsientlar bir xil ishorali va bitta koeffitsient unga qarama–qarshi ishorali bo'lsa, u holda tenglama x^0 nuqtada giperbolik tipdagi (yoki normal giperbolik tipdagi) tenglama deb, agar α_i koeffitsientlarning m tasi bir xil ishorali va $n-m$ tasi unga qarama–qarshi ishorali $(m>1,\ n-m>1)$ bo'lsa, u holda tenglama x^0 nuqtada ultragiperbolik tipdagi tenglama deb, agar α_i koeffitsientlarning hech bo'lmaganda bittasi nolga teng bo'lsa, u holda tenglama x^0 nuqtada parabolik tipdagi tenglama deb ataladi.

Kanonik formalar:

$$\Delta u+\Phi=0 \qquad \text{(elliptik tip)},$$

$$u_{x_1x_1}=\sum_{i=2}^{n}u_{x_ix_i}+\Phi \qquad \text{(giperbolik tip)},$$

$$\sum_{i=1}^{m}u_{x_ix_i}=\sum_{i=m+1}^{n}u_{x_ix_i}+\Phi \ \ (m>1, n-m>1) \ \text{(ultragiperbolik tip)},$$

$$\sum_{i=1}^{n-m}\left(\pm u_{x_ix_i}\right)+\Phi=0 \ \ (m>0) \qquad \text{(parabolik tip)}.$$

O'zgarmas koeffitsientli bo'lgan holda

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}u_{x_ix_j}+\sum_{i=1}^{n}b_iu_{x_i}+cu+f=0$$

tenglama uning aniqlanish sohasining barcha nuqtalari uchun bir vaqtda o'zgaruvchilarni chiziqli almashtirish yordamida kanonik shaklga keltiriladi. u funksiya o'rniga $u=v\cdot e^{\lambda_1x_1+\lambda_2x_2+\ldots+\lambda_nx_n}$ tenglik yordamida v yangi funksiyani kiritib va λ_i o'zgarmaslarni tegishli tanlash yordamida biz tenglamani yanada sodda shakldagi kanonik formaga keltirishimiz mumkin bo'ladi.

$n=2$ uchun

$$v_{\xi\xi}+v_{\eta\eta}+cv+f_1=0 \quad \text{(elliptik tip)},$$

$$\left.\begin{array}{l} v_{\xi\eta}+cv+f_1=0 \\ yoki \\ v_{\xi\xi}-v_{\eta\eta}+cv+f_1=0 \end{array}\right\} \quad \text{(giperbolik tip)},$$

$$v_{\xi\xi}+b_2 v_\eta+f_1=0 \quad \text{(parabolik tip)}.$$

5. Ikkinchi tartibli ikki o'zgaruvchili xususiy hosilali chiziqli differensial tenglamalarni kanonik shaklga keltirish. Quyidagi kvazichiziqli tenglamani qaraylik:

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0\ , \tag{6}$$

bunda $A,B,C\in C^2(\Omega)$.

Bu differensial tenglama

1) Agar $B^2-AC>0$ bo'lsa, u holda giperbolik tipga,

2) Agar $B^2-AC=0$ bo'lsa, u holda parabolik tipga,

3) Agar $B^2-AC<0$ bo'lsa, u holda elliptik tipga tegishli bo'ladi.

Ushbu

$$\xi=\xi(x,y),\quad \eta=\eta(x,y)$$

funksiyalar ikki marta uzluksiz differensiallanuvchi funksiyalar bo'lib, bundan tashqari Ω sohada yakobian noldan farqli, ya'ni

$$\frac{D(\xi,\eta)}{D(x,y)}=\begin{vmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{vmatrix}\neq 0$$

bo'lsin. ξ va η yangi o'zgaruvchilarga nisbatan tenglama quyidagi shaklda yoziladi:

$$\bar{A}\frac{\partial^2 u}{\partial\xi^2}+2\bar{B}\frac{\partial^2 u}{\partial\xi\partial\eta}+\bar{C}\frac{\partial^2 u}{\partial\eta^2}+\bar{F}\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)=0\,, \tag{7}$$

bunda

$$\bar{A}(\xi,\eta)=A\left(\frac{\partial\xi}{\partial x}\right)^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C\left(\frac{\partial\xi}{\partial y}\right)^2,$$

$$\bar{C}(\xi,\eta)=A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2,$$

$$\bar{B}(\xi,\eta)=A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}$$

va

$$\bar{B}^2-\bar{A}\bar{C}=\left(B^2-AC\right)\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2.$$

$\xi(x,y)$ va $\eta(x,y)$ funksiyalarni shunday tanlash mumkinki, bunda quyidagi shartlardan faqat biri bajariladi:

1) $\bar{A}=0,\ \bar{C}=0$; 2) $\bar{A}=0,\ \bar{B}=0$; 3) $\bar{A}=\bar{C},\ \bar{B}=0$.

1) $B^2-AC>0$ bo'lsin. $A\neq 0$ yoki $C\neq 0$ deb olamiz. Masalan, $A\neq 0$.

$$A\left(\frac{\partial\varphi}{\partial x}\right)^2+2B\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}+C\left(\frac{\partial\varphi}{\partial y}\right)^2=0 \tag{8}$$

tenglamani qaraylik. Bu tenglamani

$$\left[A\frac{\partial\varphi}{\partial x}+\left(B+\sqrt{B^2-AC}\right)\frac{\partial\varphi}{\partial y}\right]\times\left[A\frac{\partial\varphi}{\partial x}+\left(B-\sqrt{B^2-AC}\right)\frac{\partial\varphi}{\partial y}\right]=0$$

shaklda ham yozish mumkin. Bundan, esa

$$A\frac{\partial\varphi}{\partial x}+\left(B+\sqrt{B^2-AC}\right)\frac{\partial\varphi}{\partial y}=0, \tag{9}$$

$$A\frac{\partial\varphi}{\partial x}+\left(B-\sqrt{B^2-AC}\right)\frac{\partial\varphi}{\partial y}=0 \tag{10}$$

tenglamalar hosil bo'ladi. (9) va (10) tenglamalarni integrallash uchun ularga mos oddiy xarakteristik differensial tenglamalarni tuzamiz.

$$\frac{dx}{A}=\frac{dy}{B+\sqrt{B^2-AC}}, \quad \frac{dx}{A}=\frac{dy}{B-\sqrt{B^2-AC}},$$

yoki

$$Ady-\left(B+\sqrt{B^2-AC}\right)dx=0, \quad Ady-\left(B-\sqrt{B^2-AC}\right)dx=0,$$

yoki bitta tenglama ko'rinishida

$$A(dy)^2-2Bdydx+C(dx)^2=0$$

tenglama hosil bo'ladi. Bundan, $A(x_0,y_0)\neq 0$ bo'lgani uchun

$$\varphi_1(x,y)=const, \quad \varphi_2(x,y)=const$$

integrallar mavjudligi kelib chiqadi. (Haqiqatdan ham, o'zgarmas koeffitsientli bo'lgan holda

$$\frac{dy}{dx}=\frac{B+\sqrt{B^2-AC}}{A}, \quad \frac{dy}{dx}=\frac{B-\sqrt{B^2-AC}}{A},$$

$$y=\frac{B+\sqrt{B^2-AC}}{A}x+C_1, \quad y=\frac{B-\sqrt{B^2-AC}}{A}x+C_2).$$

$\xi=\varphi_1(x,y)$, $\eta=\varphi_2(x,y)$ deb olamiz. U holda (7) tenglamani $2\overline{B}$ ga bo'lib,

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=F_1\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)$$

tenglamani hosil qilamiz, yoki $\xi=\alpha+\beta$, $\eta=\alpha-\beta$ deb olib,

$$\frac{\partial^2 u}{\partial\alpha^2}-\frac{\partial^2 u}{\partial\beta^2}=\Phi\left(\alpha,\beta,u,\frac{\partial u}{\partial\alpha},\frac{\partial u}{\partial\beta}\right)$$

tenglikga ega bo'lamiz. Bu giperbolik tipdagi tenglamaning kanonik shaklidir.

2) $B^2-AC=0$ bo'lsin. $A\neq 0$ deb olamiz. U holda (8) tenglama quyidagi ko'rinishda bo'ladi:

$$A\frac{\partial\varphi}{\partial x}+B\frac{\partial\varphi}{\partial y}=0.$$

Bu tenglamaning $\varphi(x,y)=const$ umumiy yechimi yordamida $\xi=\varphi(x,y)$ deb olamiz va $\eta=\eta(x,y)$ sifatida esa, ikki marta uzluksiz differensiallanuvchi ixtiyoriy funksiyani $\frac{D(\xi,\eta)}{D(x,y)} \neq 0$ shart (x_0, y_0) nuqta atrofida bajariladigan qilib olamiz. $B^2 - AC = 0$ shartdan va $A\frac{\partial\varphi}{\partial x} + B\frac{\partial\varphi}{\partial y} = 0$ tenglikdan $B\frac{\partial\varphi}{\partial x} + C\frac{\partial\varphi}{\partial y} = 0$ tenglik kelib chiqadi. Shuning uchun $\overline{B} = \left(A\frac{\partial\varphi}{\partial x} + B\frac{\partial\varphi}{\partial y}\right)\frac{\partial\eta}{\partial x} + \left(B\frac{\partial\varphi}{\partial x} + C\frac{\partial\varphi}{\partial y}\right)\frac{\partial\eta}{\partial y} = 0$ bo'ladi. $\overline{A} = 0$ tenglik ham o'rinli. $\overline{C}$ koeffitsient esa, $\overline{C} = \frac{1}{A}\left(A\frac{\partial\eta}{\partial x} + B\frac{\partial\eta}{\partial y}\right)^2$ shaklga almashadi, bundan $\overline{C} \neq 0$ ekanligi kelib chiqadi. (7) tenglamada $\overline{C} \neq 0$ ga bo'lib,

$$\frac{\partial^2 u}{\partial\eta^2} = F_2\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right)$$

tenglamaga ega bo'lamiz. Bu parabolik tipdagi tenglamaning kanonik shaklidir.

3) $B^2 - AC < 0$ bo'lsin. A, B, C koeffitsientlar x va y ga bog'liq analitik funksiyalar deb olamiz. U holda

$$A\frac{\partial\varphi}{\partial x} + \left(B + \sqrt{B^2 - AC}\right)\frac{\partial\varphi}{\partial y} = 0$$

tenglama (x_0, y_0) nuqta atrofida $\varphi(x, y) = \varphi_1(x, y) + i\varphi_2(x, y)$ va shu nuqta atrofida $\left|\frac{\partial\varphi}{\partial x}\right| + \left|\frac{\partial\varphi}{\partial y}\right| \neq 0$ bo'lgan analitik yechimga ega bo'ladi. (Bunday analitik yechimning mavjudligi S.V. Kovalevskaya teoremasidan kelib chiqadi).

$$\xi = \varphi_1(x, y), \quad \eta = \varphi_2(x, y)$$

deb olamiz. $\frac{\partial(\varphi_1, \varphi_2)}{\partial(x, y)} \neq 0$ ekanligini ko'rsatish qiyin emas. Endi

$$A\left(\frac{\partial\varphi}{\partial x}\right)^2 + 2B\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y} + C\left(\frac{\partial\varphi}{\partial y}\right)^2 = 0$$

ayniyatning haqiqiy va mavhum qismlarini ajratib,

$$A\left(\frac{\partial\xi}{\partial x}\right)^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C\left(\frac{\partial\xi}{\partial y}\right)^2=A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2$$

ekanligini, ya'ni $\overline{A}=\overline{C}$ va

$$A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}=0$$

ekanligini, ya'ni $\overline{B}=0$ tengliklarni hosil qilamiz.

$$At_1^2+2Bt_1t_2+Ct_2^2\quad\left(B^2-AC<0\right)$$

kvadratik formaning aniqlanganligiga ko'ra, faqat va faqat shu holdaki, agar

$$\frac{\partial\xi}{\partial x}=\frac{\partial\xi}{\partial y}=\frac{\partial\eta}{\partial x}=\frac{\partial\eta}{\partial y}=0$$

bo'lsa, u holda $\overline{A}=\overline{C}$ nolga aylanadi. Lekin, biz $\varphi(x,y)$ yechimni bir vaqtda bu tenglikni qanoatlantirmaydigan qilib tanlaganmiz.

Shunday qilib, $\overline{A}$ ga bo'lib,

$$\frac{\partial^2u}{\partial\xi^2}+\frac{\partial^2u}{\partial\eta^2}=F_3\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)$$

tenglikga ega bo'lamiz. Bu elliptik tipdagi tenglamaning kanonik shaklidir.

6. O'zgarmas koeffitsientli ikki o'zgaruvchili ikkinchi tartibli xususiy hosilali chiziqli differensial tenglamalarni kanonik shaklga keltirishga doir misollar.

1–misol. $u_{xx}+2u_{xy}-3u_{yy}+2u_x+6u_y=0$ tenglamani kanonik ko'rinishga keltiring.

Yechish: $A=1,\ B=1,\ C=-3,\ \Delta=B^2-AC=1+3=4>0$ bo'lgani uchun, yuqoridagi tenglama giperbolik tipdagi tenglama bo'ladi. Endi esa, uning xarakteristik tenglamasini tuzamiz va uni yechamiz:

$$(dy)^2-2dydx-3(dx)^2=0,\quad y'^2-2y'-3=0,\quad y'=\frac{2\pm\sqrt{4+12}}{2}=1\pm2,$$

$$y'=3,\quad y'=-1\ y=3x-C_1,\quad y=-x+C_2,\ C_1=3x-y,\ C_2=x+y.$$

C_1, C_2 o'zgarmaslarni mos ravishda ξ,η lar bilan almashtiramiz:

$$\begin{cases} \xi = 3x - y \\ \eta = x + y \, . \end{cases}$$

u funksiyani murakkab funksiya deb qarab, birinchi va ikkinchi tartibli xususiy hosilalarni hisoblab, tenglamaga qo'yamiz:

$$u_x = u_\xi \cdot \xi_x + u_\eta \cdot \eta_x = 3u_\xi + u_\eta, \quad u_y = u_\xi \cdot \xi_y + u_\eta \cdot \eta_y = -u_\xi + u_\eta,$$

$$u_{xx} = u_{\xi\xi} \cdot (\xi_x)^2 + 2u_{\xi\eta} \cdot \xi_x \cdot \eta_x + u_{\eta\eta} \cdot (\eta_x)^2 +$$
$$+ u_\xi \cdot \xi_{xx} + u_\eta \cdot \eta_{xx} = 9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta},$$

$$u_{xy} = u_{\xi\xi} \cdot \xi_x \cdot \xi_y + u_{\xi\eta} \cdot (\xi_x \cdot \eta_y + \xi_y \cdot \eta_x) + u_{\eta\eta} \cdot \eta_x \cdot \eta_y +$$
$$+ u_\xi \cdot \xi_{xy} + u_\eta \cdot \eta_{xy} = -3u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta},$$

$$u_{yy} = u_{\xi\xi} \cdot (\xi_y)^2 + 2u_{\xi\eta} \cdot \xi_y \cdot \eta_y + u_{\eta\eta} \cdot (\eta_y)^2 +$$
$$+ u_\xi \cdot \xi_{yy} + u_\eta \cdot \eta_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}.$$

Bu yerda shuni ta'kidlab o'tamizki, agar ξ, η lar x, y larning chiziqli funksiyalari bo'lsa, u holda $\xi_{xx} = 0$, $\xi_{xy} = 0$, $\xi_{yy} = 0$, $\eta_{xx} = 0$, $\eta_{xy} = 0$, $\eta_{yy} = 0$ bo'ladi, hamda birinchi va ikkinchi tartibli hosilalar quyidagi formulalarga o'xshab ketadi:

$$u_{xx} = \left(\frac{\partial}{\partial \xi} \cdot \xi_x + \frac{\partial}{\partial \eta} \cdot \eta_x \right)^2 u = u_{\xi\xi} \cdot (\xi_x)^2 + 2u_{\xi\eta} \cdot \xi_x \cdot \eta_x + u_{\eta\eta} \cdot (\eta_x)^2,$$

$$u_{xy} = \left(\frac{\partial}{\partial \xi} \cdot \xi_x + \frac{\partial}{\partial \eta} \cdot \eta_x \right) \cdot \left(\frac{\partial}{\partial \xi} \cdot \xi_y + \frac{\partial}{\partial \eta} \cdot \eta_y \right) u =$$
$$= u_{\xi\xi} \cdot \xi_x \cdot \xi_y + u_{\xi\eta} \cdot (\xi_x \cdot \eta_y + \xi_y \cdot \eta_x) + u_{\eta\eta} \cdot \eta_x \cdot \eta_y,$$

$$u_{yy} = \left(\frac{\partial}{\partial \xi} \xi_y + \frac{\partial}{\partial \eta} \cdot \eta_y \right)^2 u = u_{\xi\xi} \cdot (\xi_y)^2 + 2u_{\xi\eta} \, \xi_y \cdot \eta_y + u_{\eta\eta} \cdot (\eta_y)^2$$

Endi topilgan ifodalarni $u_{xx} + 2u_{xy} - 3u_{yy} + 2u_x + 6u_y = 0$ tenglamaga olib borib qo'yamiz:

$$9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta} + 2 \cdot (-3u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) - 3 \cdot (u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) +$$
$$+ 2 \cdot (3u_\xi + u_\eta) + 6 \cdot (-u_\xi + u_\eta) = 0,$$

ya'ni $16u_{\xi\eta}+8u_{\eta}=0$. Bundan $u_{\xi\eta}+\frac{1}{2}u_{\eta}=0$ tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

2–misol. $u_{xx}+2u_{xy}+5u_{yy}+2u_x-3u_y=0$ tenglamani kanonik ko'rinishga keltiring.

Yechish: $A=1,\ B=1,\ C=5,\qquad \Delta=B^2-AC=1-5=-4<0$ bo'lgani uchun, yuqoridagi tenglama elliptik tipdagi tenglama bo'ladi. Endi esa, uning xarakteristik tenglamasini tuzamiz va uni yechamiz:

$$y'^2-2y'+5=0,\quad y'=\frac{2+\sqrt{4-20}}{2}=1+2i,$$

$$y=(1+2i)x-C,\qquad x-y+2xi=C.$$

Bu yerda $\begin{cases}\xi=x-y\\ \eta=2x\end{cases}$ almashtirishni bajaramiz. Shunga ko'ra,

$$u_x=u_\xi+2u_\eta,\ u_y=-u_\xi,\ u_{xx}=u_{\xi\xi}+4u_{\xi\eta}+4u_{\eta\eta},$$

$$u_{xy}=-u_{\xi\xi}-2u_{\xi\eta},\quad u_{yy}=u_{\xi\xi}$$

hosil bo'ladi. Topilgan ifodalarni tenglamaga olib borib qo'yamiz:

$$u_{\xi\xi}+4u_{\xi\eta}+4u_{\eta\eta}+2\left(-u_{\xi\xi}-2u_{\xi\eta}\right)+5u_{\xi\xi}+$$

$$+2\left(u_\xi+2u_\eta\right)-3\left(-u_\xi\right)=0,$$

ya'ni $4u_{\xi\xi}+4u_{\eta\eta}+5u_\xi+4u_\eta=0$. Bundan $u_{\xi\xi}+u_{\eta\eta}+\frac{5}{4}u_\xi+u_\eta=0$ tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

Endi esa ushbu elliptik tenglamani sodda kanonik shaklga keltiraylik, ya'ni tenglamadagi birinchi tartibli hosilalarni yo'qotamiz. Buning uchun $u(\xi,\eta)=v(\xi,\eta)\cdot e^{\lambda\xi+\mu\eta}$ almashtirish bajaramiz va birinchi va ikkinchi tartibli xususiy hosilalarni hisoblab, tenglamaga qo'yamiz:

$$u_\xi=v_\xi\cdot e^{\lambda\xi+\mu\eta}+v\cdot\lambda e^{\lambda\xi+\mu\eta},\quad u_\eta=v_\eta\cdot e^{\lambda\xi+\mu\eta}+v\cdot\mu e^{\lambda\xi+\mu\eta},$$

$$u_{\xi\xi}=v_{\xi\xi}\cdot e^{\lambda\xi+\mu\eta}+v_\xi\cdot\lambda e^{\lambda\xi+\mu\eta}+v_\xi\cdot\lambda e^{\lambda\xi+\mu\eta}+v\cdot\lambda^2e^{\lambda\xi+\mu\eta}=$$

$$=v_{\xi\xi}\cdot e^{\lambda\xi+\mu\eta}+2v_\xi\cdot\lambda e^{\lambda\xi+\mu\eta}+v\cdot\lambda^2e^{\lambda\xi+\mu\eta},$$

$$u_{\eta\eta} = v_{\eta\eta}\cdot e^{\lambda\xi+\mu\eta} + v_\eta\cdot\mu e^{\lambda\xi+\mu\eta} + v_\eta\cdot\mu e^{\lambda\xi+\mu\eta} +$$

$$+v\cdot\mu^2 e^{\lambda\xi+\mu\eta} = v_{\eta\eta}\cdot e^{\lambda\xi+\mu\eta} + 2v_\eta\cdot\mu e^{\lambda\xi+\mu\eta} + v\cdot\mu^2 e^{\lambda\xi+\mu\eta},$$

$$v_{\xi\xi}\cdot e^{\lambda\xi+\mu\eta} + v_{\eta\eta}\cdot e^{\lambda\xi+\mu\eta} + (2\lambda+\frac{5}{4})v_\xi\cdot e^{\lambda\xi+\mu\eta} +$$

$$+(2\mu+1)v_\eta\cdot e^{\lambda\xi+\mu\eta} + \left(\lambda^2+\mu^2+\frac{5}{4}\lambda+\mu\right)\cdot v\cdot e^{\lambda\xi+\mu\eta} = 0.$$

Agar $\lambda = -\frac{5}{8}$, $\mu = -\frac{1}{2}$ deb tanlasa, oxirgi tenglama quyidagi sodda kanonik shaklga keladi:

$$v_{\xi\xi} + v_{\eta\eta} - \frac{41}{64}v = 0.$$

3–misol. $u_{xx} + 4u_{xy} + 4u_{yy} + 2u_x - 3u_y = 0$ tenglamani sodda kanonik ko'rinishga keltiring.

Yechish: $A=1,\ B=2,\ C=4,\ \Delta = B^2 - AC = 4 - 1\cdot 4 = 0$ bo'lgani uchun, yuqoridagi tenglama parabolik tipdagi tenglama bo'ladi. Endi esa, uning xarakteristik tenglamasini tuzamiz va uni yechamiz:

$$y'^2 - 4y' + 4 = 0,\quad y' = \frac{4+\sqrt{16-16}}{2} = 2,$$

$$y = 2x - C,\qquad C = 2x - y.$$

Endi esa $\xi = 2x - y$ deb, η ni shunday tanlashimiz kerakki, $\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix} \neq 0$ shart bajarilishi kerak. Buning uchun $\eta = x$ deb tanlash yetarli, chunki $\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix} = \begin{vmatrix}2 & -1\\ 1 & 0\end{vmatrix} = 1 \neq 0$ bo'ladi.

Demak,

$$\begin{cases}\xi = 2x - y\\ \eta = x\end{cases}$$

almashtirishni bajaramiz. Shunga ko'ra, $u_x = 2u_\xi + u_\eta$, $u_y = -u_\xi$, $u_{xx} = 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta}$, $u_{xy} = -2u_{\xi\xi} - u_{\xi\eta}$, $u_{yy} = u_{\xi\xi}$ bo'ladi. Topilgan ifodalarni tenglamaga olib borib qo'yamiz:

$$4u_{\xi\xi}+4u_{\xi\eta}+u_{\eta\eta}+4(-2u_{\xi\xi}-u_{\xi\eta})+4u_{\xi\xi}+2(2u_{\xi}+u_{\eta})-3(-u_{\xi})=0,$$

ya'ni $u_{\eta\eta}+7u_{\xi}+2u_{\eta}=0$ tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

Endi esa ushbu parabolik tipdagi tenglamani sodda kanonik shaklga keltiraylik. Buning uchun $u(\xi,\eta)=v(\xi,\eta)\cdot e^{\lambda\xi+\mu\eta}$ almashtirish bajaramiz va yuqoridagi misoldagi kabi birinchi va ikkinchi tartibli xususiy hosilalarni hisoblab, tenglamaga qo'yamiz:

$$v_{\eta\eta}\cdot e^{\lambda\xi+\mu\eta}+7v_{\xi}\cdot e^{\lambda\xi+\mu\eta}+$$
$$+(2\mu+2)v_{\eta}\cdot e^{\lambda\xi+\mu\eta}+\left(\mu^2+7\lambda+2\mu\right)\cdot v\cdot e^{\lambda\xi+\mu\eta}=0.$$

Agar $\lambda=\frac{1}{7}$, $\mu=-1$ deb tanlasak, oxirgi tenglama quyidagi sodda kanonik shaklga keladi: $v_{\eta\eta}+7v_{\xi}=0$.

7. O'zgaruvchi koeffitsientli ikki o'zgaruvchili ikkinchi tartibli chiziqli differensial tenglamalarni kanonik shaklga keltirishga doir misollar.

1–misol. $y^2u_{xx}+2xyu_{xy}+x^2u_{yy}+2xu_x+4u=0$ tenglamani kanonik ko'rinishga keltiring.

Yechish: $A=y^2$, $B=xy$, $C=x^2$, $\Delta=x^2y^2-x^2y^2=0$ bo'lgani uchun yuqoridagi tenglama koordinata boshidan boshqa barcha joyda parabolik tipdagi tenglama bo'ladi. Endi esa xarakteristik tenglamasini tuzamiz va uni yechamiz:

$$y^2y'^2-2xyy'+x^2=0,\quad y'=\frac{2xy\pm\sqrt{4x^2y^2-4x^2y^2}}{2y^2}=\frac{x}{y},$$

$$\frac{1}{2}y^2=\frac{1}{2}x^2-\frac{1}{2}C.$$

U holda

$$\begin{cases}\xi=x^2-y^2\\ \eta=x\end{cases}$$

almashtirish bajaramiz. Shunga ko'ra,

$$u_x = 2xu_\xi + u_\eta,\ \ u_y = -2yu_\xi,\ \ u_{xx} = 4x^2u_{\xi\xi} + 4xu_{\xi\eta} + u_{\eta\eta} + 2u_\xi\ ,$$

$$u_{xy} = -4xyu_{\xi\xi} - 2yu_{\xi\eta},\ \ u_{yy} = 4y^2u_{\xi\xi} - 2u_\xi\ .$$

Bu topilgan ifodalarnini tenglamaga olib borib qo‘yamiz:

$$y^2\left(4x^2u_{\xi\xi} + 4xu_{\xi\eta} + u_{\eta\eta} + 2u_\xi\right) + 2xy\left(-4xyu_{\xi\xi} - 2yu_{\xi\eta}\right) +$$

$$+x^2\left(4y^2u_{\xi\xi} - 2u_\xi\right) + 2x\left(2xu_\xi + u_\eta\right) + 4u = 0.$$

Natijada

$$y^2u_{\eta\eta} + 2\left(x^2 + y^2\right)u_\xi + 2xu_\eta + 4u = 0\,,$$

yoki

$$\left(\eta^2 - \xi\right)u_{\eta\eta} + 2\left(2\eta^2 - \xi\right)u_\xi + 2\eta u_\eta + 4u = 0$$

tenglamaga ega bo‘lamiz. Bundan esa,

$$u_{\eta\eta} + \frac{2\left(2\eta^2 - \xi\right)}{\eta^2 - \xi}u_\xi + \frac{2\eta}{\eta^2 - \xi}u_\eta + \frac{4}{\eta^2 - \xi}u = 0$$

tenglama hosil bo‘ladi. Ushbu tenglama berilgan tenglamaning kanonik ko‘rinishi bo‘ladi.

2–misol. $yu_{xx} + u_{yy} = 0$ tenglamani kanonik ko‘rinishga keltiring.

Yechish: Bu tenglama Trikomi tenglamasi deb ataladi va unda $A = y,\ B = 0,\ C = 1,\ \ \Delta = -y$.

Agar

1) $y < 0$ bo‘lsa berilgan tenglama giperbolik tipda,

2) $y = 0$ bo‘lsa berilgan tenglama parabolik tipda,

3) $y > 0$ bo‘lsa berilgan tenglama elliptik tipda bo‘ladi.

Trikomi tenglamasi gaz dinamikasi uchun muhim bo‘lib, giperbolik sohada bu tenglama tovush tezligidan yuqori harakatga mos va elliptik sohada esa, bu tenglama tovush tezligigacha bo‘lgan harakatga mos keladi.

1) $y < 0$ bo‘lsin. U holda xarakteristik tenglama quyidagicha bo‘ladi:

$$yy'^2 + 1 = 0,\ \sqrt{-y}\cdot y' \pm 1 = 0,\ \sqrt{-y}\cdot dy \pm dx = 0.$$

Bundan esa,

$$-\frac{2}{3}\sqrt{(-y)^3}-x=C_1,\quad -\frac{2}{3}\sqrt{(-y)^3}+x=C_2$$

bo'ladi. U holda quyidagicha almashtirish bajaramiz:

$$\begin{cases}\xi=x+\dfrac{2}{3}(-y)^{\frac{3}{2}}\\ \eta=x-\dfrac{2}{3}(-y)^{\frac{3}{2}}.\end{cases}$$

Shunga ko'ra,

$$u_x=u_\xi+u_\eta,\quad u_y=-(-y)^{\frac{1}{2}}u_\xi+(-y)^{\frac{1}{2}}u_\eta,\quad u_{xx}=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta},$$

$$u_{yy}=\frac{1}{2\sqrt{-y}}u_\xi-(-y)^{\frac{1}{2}}\left(-(-y)^{\frac{1}{2}}u_{\xi\xi}+(-y)^{\frac{1}{2}}u_{\xi\eta}\right)-\frac{1}{2\sqrt{-y}}u_\eta+$$

$$+(-y)^{\frac{1}{2}}\left(-(-y)^{\frac{1}{2}}u_{\xi\eta}+(-y)^{\frac{1}{2}}u_{\eta\eta}\right).$$

Bu topilgan ifodalarni tenglamaga olib borib qo'yamiz:

$$y(u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta})+\frac{1}{2\sqrt{-y}}(u_\xi-u_\eta)+y(-u_{\xi\xi}+u_{\xi\eta})-$$

$$-y(-u_{\xi\eta}+u_{\eta\eta})=0\ ,$$

$$4yu_{\xi\eta}+\frac{1}{2\sqrt{-y}}(u_\xi-u_\eta)=0,\quad u_{\xi\eta}+\frac{1}{8y\sqrt{-y}}(u_\xi-u_\eta)=0,$$

$$u_{\xi\eta}-\frac{1}{6(\xi-\eta)}(u_\xi-u_\eta)=0\quad(\xi>\eta)$$

tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

2) $y=0$ bo'lsin. U holda xarakteristik tenglama quyidagicha bo'ladi: $(dx)^2=0$, $dx=0$. Bundan esa,

$$x=C$$

bo'ladi. Shuning uchun

$$\begin{cases} \xi = x \\ \eta = y \end{cases}$$

almashtirish bajaramiz va $y = 0$ bo'lgani uchun $u_{yy} = 0$ tenglamaga ega bo'lamiz.

3) $y > 0$ bo'lsin. U holda xarakteristik tenglama quyidagicha bo'ladi:

$$y(dy)^2 + (dx)^2 = 0, \left(\sqrt{y}dy\right)^2 - (idx)^2 = 0, \sqrt{y}dy \pm idx = 0 .$$

Bundan esa,

$$\frac{2}{3} y^{\frac{3}{2}} \pm ix = C$$

bo'ladi. U holda quyidagicha almashtirish bajaramiz :

$$\begin{cases} \xi = \frac{2}{3} y^{\frac{3}{2}} \\ \eta = x \end{cases} .$$

Shunga ko'ra,

$$u_x = u_\eta, \quad u_y = y^{\frac{1}{2}} u_\xi, \quad u_{xx} = u_{\eta\eta},$$

$$u_{yy} = \frac{1}{2\sqrt{y}} u_\xi + y^{\frac{1}{2}} \left(y^{\frac{1}{2}} u_{\xi\xi} \right) = \frac{1}{2\sqrt{y}} u_\xi + y u_{\xi\xi}.$$

Bu topilgan ifodalarnini tenglamaga olib borib qo'yamiz. Natijada

$$y u_{\eta\eta} + y u_{\xi\xi} + \frac{1}{2\sqrt{y}} u_\xi = 0,$$

yoki $u_{\eta\eta} + u_{\xi\xi} + \frac{1}{2y\sqrt{y}} u_\xi = 0$, hamda $\xi = \frac{2}{3} y^{\frac{3}{2}}$ ekanligidan

$$u_{\eta\eta} + u_{\xi\xi} + \frac{1}{3\xi} u_\xi = 0$$

kanonik shakldagi tenglamaga ega bo'lamiz.

8. Ko'p o'zgaruvchili ikkinchi tartibli chiziqli differensial tenglamalarni kanonik shaklga keltirish. Bizga ushbu

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}+c(x)u=f(x)$$

ikkinchi tartibli xususiy hosilali differensial tenglama berilgan bo'lsin. Bu tenglamani kanonik shaklga keltirish masalasini qaraylik. Berilgan tenglamani har bir tayinlangan $x=x_0$ nuqtada maxsus bo'lmagan

$\xi = B^T x$, bunda $x=\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, chiziqli almashtirish yordamida kanonik

shaklga keltirish mumkin. Bu yerda B — shunday matritsaki, $\lambda = B\mu$

chiziqli almashtirish $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_o)\lambda_i\lambda_j$ kvadratik formani kanonik

ko'rinishga keltiradi.

Har qanday kvadratik formani kanonik ko'rinishga keltirishning turli usullari mavjud, masalan shunday usullardan biri to'la kvadrat ajratish usulidir.

1–misol. $u_{xx}+2u_{xy}-2u_{xz}+2u_{yy}+6u_{zz}=0$ tenglamani kanonik ko'rinishga keltiring.

Yechish: Berilgan tenglamaga mos kvadratik formani tuzamiz va uni kanonik shaklga keltiramiz:

$$\lambda_1^2+2\lambda_1\lambda_2-2\lambda_1\lambda_3+2\lambda_2^2+6\lambda_3^2=(\lambda_1+\lambda_2-\lambda_3)^2+\lambda_2^2+2\lambda_2\lambda_3+5\lambda_3^2=$$
$$=(\lambda_1+\lambda_2-\lambda_3)^2+(\lambda_2+\lambda_3)^2+4\lambda_3^2=\mu_1^2+\mu_2^2+\mu_3^2.$$

Berilgan tenglama elliptik tipda ekanligi ko'rinib turibdi, bu yerda

$$\begin{cases}\mu_1=\lambda_1+\lambda_2-\lambda_3\\ \mu_2=\lambda_2+\lambda_3\\ \mu_3=2\lambda_3\end{cases}.$$

Bundan esa quyidagi tengliklarga ega bo'lamiz:

$$\begin{cases} \lambda_1 = \mu_1 - \mu_2 + \mu_3 \\ \lambda_2 = \mu_2 - \frac{1}{2}\mu_3 \\ \lambda_3 = \frac{1}{2}\mu_3 \end{cases}.$$

Bu almashtirishning

$$B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

matritsasini hosil qilamiz va uni transponirlaymiz. U holda

$$B^T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

bo'ladi. Endi quyidagicha almashtirish bajaramiz:

$$\xi = B^T X, \quad \text{bunda} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

ya'ni

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x + y \\ x - \frac{1}{2}y + \frac{1}{2}z \end{pmatrix},$$

yoki

$$\begin{cases} \xi_1 = x \\ \xi_2 = -x + y \\ \xi_3 = x - \frac{1}{2}y + \frac{1}{2}z. \end{cases}$$

Berilgan almashtirish chiziqli ekanligini hisobga olib,

$$u_x = u_{\xi_1} - u_{\xi_2} + u_{\xi_3}, \quad u_y = u_{\xi_2} - \frac{1}{2}u_{\xi_3}, \quad u_z = \frac{1}{2}u_{\xi_3},$$

$$u_{xx} = u_{\xi_1\xi_1} - 2u_{\xi_1\xi_2} + 2u_{\xi_1\xi_3} - 2u_{\xi_2\xi_3} + u_{\xi_2\xi_2} + u_{\xi_3\xi_3},$$

$$u_{xy} = u_{\xi_1\xi_2} - \frac{1}{2}u_{\xi_1\xi_3} - u_{\xi_2\xi_2} + \frac{3}{2}u_{\xi_2\xi_3} - \frac{1}{2}u_{\xi_3\xi_3},$$

$$u_{yy} = u_{\xi_2\xi_2} - u_{\xi_2\xi_3} + \frac{1}{4}u_{\xi_3\xi_3},$$

$$u_{xz} = \frac{1}{2}u_{\xi_1\xi_3} - \frac{1}{2}u_{\xi_2\xi_3} + \frac{1}{2}u_{\xi_3\xi_3}, \qquad u_{zz} = \frac{1}{4}u_{\xi_3\xi_3}$$

tengliklarga ega bo'lamiz. Bu topilgan ifodalarni tenglamaga olib borib qo'yamiz:

$$u_{\xi_1\xi_1} - 2u_{\xi_1\xi_2} + 2u_{\xi_1\xi_3} - 2u_{\xi_2\xi_3} + u_{\xi_2\xi_2} + u_{\xi_3\xi_3} +$$

$$+2\left(u_{\xi_1\xi_2} - \frac{1}{2}u_{\xi_1\xi_3} - u_{\xi_2\xi_2} + \frac{3}{2}u_{\xi_2\xi_3} - \frac{1}{2}u_{\xi_3\xi_3}\right) - 2\left(\frac{1}{2}u_{\xi_1\xi_3} - \frac{1}{2}u_{\xi_2\xi_3} + \frac{1}{2}u_{\xi_3\xi_3}\right) +$$

$$+2\left(u_{\xi_2\xi_2} - u_{\xi_2\xi_3} + \frac{1}{4}u_{\xi_3\xi_3}\right) + 6 \cdot \frac{1}{4}u_{\xi_3\xi_3} = 0.$$

Natijada $u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + u_{\xi_3\xi_3} = 0$ tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

2–misol. $4u_{xx} - 4u_{xy} - 2u_{yz} + u_y + u_z = 0$ tenglamani kanonik ko'rinishga keltiring.

Yechish: Berilgan tenglamaga mos kvadratik formani tuzamiz va uni kanonik shaklga keltiramiz:

$$4\lambda_1^2 - 4\lambda_1\lambda_2 - 2\lambda_2\lambda_3 = (2\lambda_1 - \lambda_2)^2 - \lambda_2^2 - 2\lambda_2\lambda_3 =$$
$$= (2\lambda_1 - \lambda_2)^2 - (\lambda_2 + \lambda_3)^2 + \lambda_3^2 = \mu_1^2 - \mu_2^2 + \mu_3^2.$$

Berilgan tenglama giperbolik tipda ekanligi ko'rinib turibdi, bu yerda

$$\begin{cases} \mu_1 = 2\lambda_1 - \lambda_2 \\ \mu_2 = \lambda_2 + \lambda_3 \\ \mu_3 = \lambda_3. \end{cases}$$

Bundan esa quyidagi tengliklarga ega bo'lamiz:

$$\begin{cases} \lambda_1 = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 - \frac{1}{2}\mu_3 \\ \lambda_2 = \mu_2 - \mu_3 \\ \lambda_3 = \mu_3. \end{cases}$$

Bu almashtirishning B matritsani tuzamiz va uni transponirlaymiz. U holda

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^T = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -1 & 1 \end{pmatrix}$$

bo'ladi. Endi quyidagicha almashtirish bajaramiz: $\xi = B^T X$, ya'ni

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{x}{2} \\ \frac{x}{2} + y \\ -\frac{1}{2}x - y + z \end{pmatrix},$$

yoki

$$\begin{cases} \xi_1 = \frac{x}{2} \\ \xi_2 = \frac{x}{2} + y \\ \xi_3 = -\frac{1}{2}x - y + z. \end{cases}$$

Berilgan almashtirish chiziqli ekanligini hisobga olib,

$$u_x=\frac{1}{2}u_{\xi_1}+\frac{1}{2}u_{\xi_2}-\frac{1}{2}u_{\xi_3},\quad u_y=u_{\xi_2}-u_{\xi_3},\quad u_z=u_{\xi_3},$$

$$u_{xx}=\frac{1}{4}u_{\xi_1\xi_1}+\frac{1}{2}u_{\xi_1\xi_2}-\frac{1}{2}u_{\xi_1\xi_3}-\frac{1}{2}u_{\xi_2\xi_3}+\frac{1}{4}u_{\xi_2\xi_2}+\frac{1}{4}u_{\xi_3\xi_3},$$

$$u_{xy}=\frac{1}{2}u_{\xi_1\xi_2}-\frac{1}{2}u_{\xi_1\xi_3}+\frac{1}{2}u_{\xi_2\xi_2}-u_{\xi_2\xi_3}+\frac{1}{2}u_{\xi_3\xi_3},\quad u_{yz}=u_{\xi_2\xi_3}-u_{\xi_3\xi_3}$$

tengliklarga ega bo'lamiz va topilgan ifodalarni

$$4u_{xx}-4u_{xy}-2u_{yz}+u_y+u_z=0$$

tenglamaga olib borib qo'yamiz. U holda

$$u_{\xi_1\xi_1}+2u_{\xi_1\xi_2}-2u_{\xi_1\xi_3}-2u_{\xi_2\xi_3}+u_{\xi_2\xi_2}+u_{\xi_3\xi_3}-2u_{\xi_1\xi_2}+2u_{\xi_1\xi_3}-$$
$$-2u_{\xi_2\xi_2}+4u_{\xi_2\xi_3}-2u_{\xi_3\xi_3}-2u_{\xi_2\xi_3}+2u_{\xi_3\xi_3}+u_{\xi_2}-u_{\xi_3}+u_{\xi_3}=0.$$

Bundan esa,

$$u_{\xi_1\xi_1}-u_{\xi_2\xi_2}+u_{\xi_3\xi_3}+u_{\xi_2}=0$$

tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

3–misol. $u_{xy}-u_{xz}+u_x+u_y-u_z=0$ tenglamani kanonik ko'rinishga keltiring.

Yechish: Berilgan tenglamaga mos kvadratik formani tuzamiz va uni kanonik shaklga keltiramiz:

$$K(\lambda)=\lambda_1\lambda_2-\lambda_1\lambda_3$$

kvadratik formani kanonik ko'rinishga keltirish uchun

$$\begin{cases}\lambda_1=\eta_1+\eta_2\\ \lambda_2=\eta_1-\eta_2\\ \lambda_3=\eta_3\end{cases}$$

almashtirish bajaramiz va quyidagi kvadratik formaga ega bo'lamiz:

$$K_1(\eta)=\eta_1^2-\eta_2^2-\eta_1\eta_3-\eta_2\eta_3.$$

Ushbu kvadratik formani kanonik shaklga keltiramiz:

$$K_1(\eta)=\eta_1^2-\eta_2^2-\eta_1\eta_3-\eta_2\eta_3=\left(\eta_1-\frac{1}{2}\eta_3\right)^2-\eta_2^2-\eta_2\eta_3-$$

$$-\frac{1}{4}\eta_3^2=\left(\eta_1-\frac{1}{2}\eta_3\right)^2-\left(\eta_2+\frac{1}{2}\eta_3\right)^2=\mu_1^2-\mu_2^2.$$

Berilgan tenglama parabolik tipda ekanligi ko'rinib turibdi, bu yerda

$$\begin{cases}\mu_1=\eta_1-\frac{1}{2}\eta_3\\ \mu_2=\eta_2+\frac{1}{2}\eta_3\end{cases}$$

va μ_3 o'zgaruvchini shunday tanlaymizki, hosil bo'lgan matritsaning determinanti noldan farqli bo'lsin, masalan, $\mu_3=\eta_3$. U holda quyidagi almashtirishga ega bo'lamiz:

$$\begin{cases}\mu_1=\eta_1-\frac{1}{2}\eta_3\\ \mu_2=\eta_2+\frac{1}{2}\eta_3\\ \mu_3=\eta_3\end{cases}.$$

Bundan esa quyidagi tengliklarga ega bo'lamiz:

$$\begin{cases}\lambda_1=\mu_1+\mu_2\\ \lambda_2=\mu_1-\mu_2+\mu_3\\ \lambda_3=\mu_3\end{cases}.$$

Bu almashtirishning B matritsasini tuzamiz va uni transponirlaymiz. U holda

$$B=\begin{pmatrix}1&1&0\\1&-1&1\\0&0&1\end{pmatrix},\qquad B^T=\begin{pmatrix}1&1&0\\1&-1&0\\0&1&1\end{pmatrix}$$

bo'ladi. Endi quyidagicha almashtirish bajaramiz: $\xi=B^TX$, ya'ni

$$\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}=\begin{pmatrix}1&1&0\\1&-1&0\\0&1&1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}x+y\\ x-y\\ y+z\end{pmatrix},$$

yoki

$$\begin{cases} \xi_1 = x + y \\ \xi_2 = x - y \\ \xi_3 = y + z\ . \end{cases}$$

Berilgan almashtirish chiziqli ekanligini hisobga olib,

$$u_x = u_{\xi_1} + u_{\xi_2},\quad u_y = u_{\xi_1} - u_{\xi_2} + u_{\xi_3},\quad u_z = u_{\xi_3},$$

$$u_{xy} = u_{\xi_1\xi_1} - u_{\xi_1\xi_2} + u_{\xi_1\xi_3} + u_{\xi_1\xi_2} - u_{\xi_2\xi_2} + u_{\xi_2\xi_3} =$$

$$= u_{\xi_1\xi_1} + u_{\xi_1\xi_3} - u_{\xi_2\xi_2} + u_{\xi_2\xi_3},\qquad u_{xz} = u_{\xi_1\xi_3} + u_{\xi_2\xi_3}$$

tengliklarga ega bo'lamiz va topilgan ifodalarni tenglamaga olib borib qo'yamiz:

$$u_{\xi_1\xi_1} + u_{\xi_1\xi_3} - u_{\xi_2\xi_2} + u_{\xi_2\xi_3} - u_{\xi_1\xi_3} - u_{\xi_2\xi_3} +$$
$$+ u_{\xi_1} + u_{\xi_2} + u_{\xi_1} - u_{\xi_2} + u_{\xi_3} - u_{\xi_3} = 0\,.$$

Bundan esa,

$$u_{\xi_1\xi_1} - u_{\xi_2\xi_2} + 2u_{\xi_1} = 0$$

tenglamaga ega bo'lamiz. Ushbu tenglama berilgan tenglamaning kanonik ko'rinishi bo'ladi.

$G \subset R^n$ sohada

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) u_{x_i x_j} + \Phi(x, u, grad u) = 0,\ x \in G$$

tenglamani $A(x) = \left\| a_{ij}(x) \right\| \neq 0, x \in G$ haqiqiy simmetrik matritsa bo'lgan holda qaraymiz. Ixtiyoriy $x^0 \in G$ nuqtada $A(x^0)$ matritsaning

$$(A(x^0)y, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x^0) y_i y_j$$

kvadratik formasini $y = B\eta,\ y \in R^n,\ \eta \in R^n,\ B = B(x^0)$ maxsusmas almashtirish yordamida η vektor koordinatalari kvadratlarining algebraik yig'indisini ifoda qiluvchi kanonik ko'rinishga keltirish mumkin. Shu bilan birga, $n_+ = n_+(x^0)$ orqali 1 ga teng koeffitsientlar sonini, $n_- = n_-(x^0)$ orqali -1 ga teng koeffitsientlar sonini, $n_0 = n_0(x^0)$ orqali 0 ga teng koeffitsientlar sonini, ya'ni qatnashmayotgan koeffitsientlar sonini belgilasak, u holda $n_+ + n_- + n_0 = n$ bo'ladi. Inertsiya qonuniga ko'ra, bu

n_+, n_-, n_0 sonlar $y = B\eta$ maxsus bo'lmagan almashtirishning tanlanishiga bog'liq bo'lmaydi.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)u_{x_ix_j} + \Phi(x,u,gradu) = 0, \ x \in G$$

tenglama uchun $x^0 \in G$ nuqtada:

- agar $n_0 = 0$, $n_+ = n$ yoki $n_0 = 0$, $n_- = n$ bo'lsa, u holda bu tenglama elliptik tipdagi tenglamaga tegishli;
- agar $n_0 = 0$, $n_+ = n-1$, $n_- = 1$ yoki $n_0 = 0$, $n_+ = 1$ $n_- = n-1$ bo'lsa, u holda bu tenglama giperbolik tipdagi tenglamaga tegishli;
- agar $n_0 = 0$, $1 < n_+ < n-1$, $1 < n_- < n-1$ bo'lsa, u holda bu tenglama ultragiperbolik tipdagi tenglamaga tegishli;
- agar $n_0 > 0$ bo'lsa, u holda bu tenglama parabolik tipdagi tenglamaga tegishli bo'ladi.

Agar $G_1 \subset G$ bo'lgan to'plamning har bir $x^0 \in G_1$ nuqtasida $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)u_{x_ix_j} + \Phi(x,u,gradu) = 0$ tenglama:

- elliptik tipdagi tenglamaga tegishli bo'lsa, u holda bu tenglama G_1 to'plamda elliptik tipdagi tenglamaga tegishli;
- giperbolik tipdagi tenglamaga tegishli bo'lsa, u holda bu tenglama G_1 to'plamda giperbolik tipdagi tenglamaga tegishli;
- parabolik tipdagi tenglamaga tegishli bo'lsa, u holda bu tenglama G_1 to'plamda parabolik tipdagi tenglamaga tegishli deb aytiladi.

$\xi = B^* x$, $x \in R^n$, $\xi \in R^n$ almashtirish berilgan tenglamani $x^0 \in G$ nuqtada kanonik ko'rinishga keltiradi.

Bunday almashtirishni berilgan tenglamaning $A(x^0)$ matritsasining xos vektorlari yordamida bazis qurish va bu bazisda $A(x^0)$ matritsani diagonal shaklga keltirish va tenglamani kanonik ko'rinishga keltirish mumkin bo'ladi. Bu usul bilan 3–misoldagi

$$u_{xy} - u_{xz} + u_x + u_y - u_z = 0$$

tenglamaning tipini aniqlaymiz va uni kanonik ko'rinishga keltiramiz. Berilgan tenglamaning $A(x,y,z)$ matritsasining ko'rinishi

$$A(x,y,z)=A=\begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

shaklga ega bo'ladi. A matritsaning xos sonlarini topamiz:

$$\det(A-\lambda E)=-\lambda^3+\frac{1}{2}\lambda=-\lambda\left(\lambda^2-\frac{1}{2}\right)=0.$$

Bundan $\lambda_1=0,\ \lambda_2=\frac{1}{\sqrt{2}},\ \lambda_3=-\frac{1}{\sqrt{2}}$ ekanligi kelib chiqadi va shunga ko'ra, berilgan tenglama $(\lambda_1=0)$ parabolik tipda bo'ladi.

A matritsaning xos vektorlarini topamiz:

a) $\lambda_1=0$ bo'lsin. Bu holda

$$\begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ x \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

yoki

$$\frac{1}{2}y-\frac{1}{2}z=0,\ \frac{1}{2}x=0,-\frac{1}{2}x=0,$$

ya'ni $x=0,\ y=z$. Shunga ko'ra, $\lambda_1=0$ xos songa mos vektor $h_1=(0,1,1)^T$ bo'ladi;

b) $\lambda_2=\frac{1}{\sqrt{2}}$ bo'lsin. Bu holda

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

yoki

$$-\frac{1}{\sqrt{2}}x+\frac{1}{2}y-\frac{1}{2}z=0,\ \frac{1}{2}x-\frac{1}{\sqrt{2}}y=0,-\frac{1}{2}x-\frac{1}{\sqrt{2}}z=0,$$

yoki

$$-\sqrt{2}x+y-z=0,\ -\frac{1}{\sqrt{2}}x+y=0,-\frac{1}{\sqrt{2}}x-z=0,$$

ya'ni $y=\frac{1}{\sqrt{2}}x,\ \ z=-\frac{1}{\sqrt{2}}x$. Shunga ko'ra, $\lambda_2=\frac{1}{\sqrt{2}}$ xos songa mos vektor $h_2=\left(1,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)^T$ bo'ladi;

v) $\lambda_3=-\frac{1}{\sqrt{2}}$ bo'lsin. Bu holda

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

yoki

$$\frac{1}{\sqrt{2}}x+\frac{1}{2}y-\frac{1}{2}z=0,\ \frac{1}{2}x+\frac{1}{\sqrt{2}}y=0,-\frac{1}{2}x+\frac{1}{\sqrt{2}}z=0,$$

yoki

$$\sqrt{2}x+y-z=0,\ \ \frac{1}{\sqrt{2}}x+y=0,\ \ \frac{1}{\sqrt{2}}x-z=0,$$

ya'ni $y=-\frac{1}{\sqrt{2}}x,\ z=\frac{1}{\sqrt{2}}x$. Shunga ko'ra, $\lambda_3=-\frac{1}{\sqrt{2}}$ xos songa mos vektor $h_3=\left(1,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)^T$.

Ushbu $h_1=(0,1,1)^T$, $h_2=\left(1,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)^T$, $h_3=\left(1,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)^T$ vektorlardan tuzilgan bazisni qaraymiz.

Nuqtaning yangi bazisdagi koordinatalari (ξ,η,ζ) bo'lsin. U holda

$$\begin{cases}\xi=y+z\\ \eta=x+\frac{1}{\sqrt{2}}y-\frac{1}{\sqrt{2}}z\\ \zeta=x-\frac{1}{\sqrt{2}}y+\frac{1}{\sqrt{2}}z\end{cases}$$

va A matritsa bu bazisda

$$\begin{pmatrix}0&0&0\\ 0&\frac{1}{\sqrt{2}}&0\\ 0&0&-\frac{1}{\sqrt{2}}\end{pmatrix}.$$

diagonal shaklga keltiriladi. Bu (ξ,η,ζ) o'zgaruvchilarga nisbatan kvadratik forma

$$0\cdot\xi^2+\frac{1}{\sqrt{2}}\eta^2-\frac{1}{\sqrt{2}}\zeta^2$$

shaklga keltiriladi. Bu (ξ,η,ζ) o'zgaruvchilarda berilgan tenglamani yozamiz. Yangi o'zgaruvchilarga nisbatan berilgan tengliklarga ko'ra,

$$u_x=u_\eta+u_\zeta,\quad u_y=u_\xi+\frac{1}{\sqrt{2}}u_\eta-\frac{1}{\sqrt{2}}u_\zeta,\quad u_z=u_\xi-\frac{1}{\sqrt{2}}u_\eta+\frac{1}{\sqrt{2}}u_\zeta,$$

$$u_{xy} = u_{\xi\eta} + u_{\xi\zeta} + \frac{1}{\sqrt{2}}u_{\eta\eta} - \frac{1}{\sqrt{2}}u_{\zeta\zeta}, \quad u_{xz} = u_{\xi\eta} + u_{\xi\zeta} - \frac{1}{\sqrt{2}}u_{\eta\eta} + \frac{1}{\sqrt{2}}u_{\zeta\zeta}$$

xususiy hosilalarni topamiz. Bundan, o'xshash hadlarni keltirib,

$$\sqrt{2}u_{\eta\eta} - \sqrt{2}u_{\zeta\zeta} + (1+\sqrt{2})u_{\eta} + (1-\sqrt{2})u_{\zeta} = 0$$

yo'ki,

$$0 \cdot u_{\xi\xi} + 1 \cdot u_{\eta\eta} - 1 \cdot u_{\zeta\zeta} + \frac{1+\sqrt{2}}{\sqrt{2}}u_{\eta} + \frac{1-\sqrt{2}}{\sqrt{2}}u_{\zeta} = 0$$

tenglamaga ega bolamiz.

9. Chiziqli bo'lmagan xususiy hosilali differensial tenglamani uning berilgan yechimi bo'ylab sinflarga ajratish. Bizga ushbu

$$F\left(x,\ldots,\frac{\partial^k u}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}},\ldots\right) = 0 \qquad (11)$$

shakldagi $u(x) = u(x_1, x_2, \ldots, x_n)$, $x \in \Omega$ noma'lum funksiyaga nisbatan $m-$tartibli xususiy hosilali differensial tenglama berilgan bo'lsin.

Chiziqli bo'lmagan $m-$ tartibli xususiy hosilali (11) differensial tenglamani ham

$$K(\xi_1,\ldots,\xi_n) = \sum_{|\alpha|=m} \frac{\partial F}{\partial p_{\alpha_1\ldots\alpha_n}} \xi_1^{\alpha_1}\ldots\xi_n^{\alpha_n}$$

xarakteristik forma orqali sinflarga ajratiladi. Lekin, xarakteristik formaning koeffitsientlari bu holda $x \in \Omega$ nuqtadan tashqari izlanayotgan yechim va uning xususiy hosilalariga ham bog'liq bo'ladi. Bu holda $m-$tartibli xususiy hosilali differensial tenglamani berilgan yechim uchungina sinflarga ajratiladi.

1–misol. Quyidagi tenglamani berilgan yechim bo'ylab tipini aniqlang: $u_{xx}^2 + (u_{xx} - 2)u_{xy} - u_{yy}^2 = 0$, $u = x^2 + y^2$.

Yechish: Bu tenglamaga mos,

$$K = F_{u_{xx}}\lambda_1^2 + F_{u_{xy}}\lambda_1\lambda_2 + F_{u_{yy}}\lambda_2^2$$

kvadratik formani qaraylik. Berilgan yechim uchun,

$$K = \left(2u_{xx} + u_{xy}\right)\lambda_1^2 + \left(u_{xx} - 2\right)\lambda_1\lambda_2 - 2u_{yy}\lambda_2^2,$$

$$u_{xx} = 2, \quad u_{xy} = 0, \quad u_{yy} = 2$$

ekanliginni hisobga olsa, $K=4\lambda_1^2-4\lambda_2^2$ bo'ladi. Agar $\lambda_1=\frac{1}{2}\xi$, $\lambda_2=\frac{1}{2}\eta$ almashtirish bajarsa, $K=\xi^2-\eta^2$ formaga ega bo'lamiz. Demak, berilgan tenglama uning berilgan yechimi bo'ylab giperbolik tipdagi tenglama bo'ladi.

2–misol. Quyidagi tenglamani berilgan yechim bo'ylab tipini aniqlang: $u_{xx}^2+(u_{xx}-2)u_{xy}+u_{yy}^2+4u_{yy}+4=0$, $u=2xy$.

Yechish: Bu tenglamaga mos,

$$K=\left(2u_{xx}+u_{xy}\right)\lambda_1^2+\left(u_{xx}-2\right)\lambda_1\lambda_2+(2u_{yy}+4)\lambda_2^2$$

va $u_{xx}=0$, $u_{xy}=2$, $u_{yy}=0$ ekanligini hisobga olsak, u holda $K=2\lambda_1^2-2\lambda_1\lambda_2+4\lambda_2^2=2\left(\lambda_1-\frac{1}{2}\lambda_2\right)^2+3{,}5\lambda_2^2$ bo'ladi. Agar $\lambda_1=\frac{1}{\sqrt{2}}\xi+\sqrt{\frac{1}{14}}\eta$, $\lambda_2=\sqrt{\frac{2}{7}}\eta$ almashtirish bajarsak, $K=\xi^2+\eta^2$ formaga ega bo'lamiz. Demak, berilgan tenglama uning berilgan yechimi bo'ylab elliptik tipdagi tenglama bo'ladi.

3-misol. Quyidagi tenglamani berilgan yechim bo'ylab tipini aniqlang: $u_{xx}+u_{xy}u_{yy}+u_{yy}^2-4u_{yy}=0$, $u=2y^2$.

Yechish: Bu tenglamaga mos,

$$K=\lambda_1^2+u_{yy}\lambda_1\lambda_2+(u_{xy}+2u_{yy}-4)\lambda_2^2$$

va $u_{xx}=0$, $u_{xy}=0$, $u_{yy}=4$ ekanligini hisobga olsak, u holda $K=\lambda_1^2+4\lambda_1\lambda_2+4\lambda_2^2=\left(\lambda_1+2\lambda_2\right)^2$ bo'ladi. Agar $\lambda_1=\xi-2\eta$, $\lambda_2=\eta$ almashtirish bajarsak, $K=\xi^2$ formaga ega bo'lamiz. Demak, berilgan tenglama uning berilgan yechimi bo'ylab parabolik tipdagi tenglama bo'ladi.

4-misol. Quyidagi tenglamani berilgan $u(x,y)$ yechim bo'ylab tipini aniqlang: $u_{xx}^2-4u_{xy}+u_{yy}^2=0$, $u=(x+y)^2$, $u=x$, $u=x^2+\frac{y^2}{4}+\frac{17}{16}xy$.

Yechish: Bu tenglamaga mos, $F = u_{xx}^2 - 4u_{xy} + u_{yy}^2$, $\dfrac{\partial F}{\partial u_{xx}} = 2u_{xx}$,

$\dfrac{\partial F}{\partial u_{xy}} = -4$, $\dfrac{\partial F}{\partial u_{yy}} = 2u_{yy}$ bo'lgani uchun xarakteristik ko'phad

$$K = \frac{\partial F}{\partial u_{xx}}\lambda_1^2 + \frac{\partial F}{\partial u_{xy}}\lambda_1\lambda_2 + \frac{\partial F}{\partial u_{yy}}\lambda_2^2 = 2u_{xx}\lambda_1^2 - 4\lambda_1\lambda_2 + 2u_{yy}\lambda_2^2$$

ko'rinishda bo'ladi. Shunga ko'ra,

1) agar $u = (x+y)^2$ bo'lsa, u holda $u_{xx} = 2$, $u_{xy} = 0$, $u_{yy} = 2$ ekanligini hisobga olsak, u holda $K = 4\lambda_1^2 - 4\lambda_1\lambda_2 + 4\lambda_2^2$ bo'ladi, bunda $A = 4$, $B = -2$, $C = 4$ va $B^2 - AC = 4 - 16 = -12 < 0$ ekanligidan tenglamaning berilgan $u = (x+y)^2$ yechimi bo'ylab tipi elliptik tipdagi tenglama bo'ladi.

2) agar $u = x$ bo'lsa, u holda $u_{xx} = 0$, $u_{xy} = 0$, $u_{yy} = 0$ ekanligini hisobga olsak, u holda $K = -4\lambda_1\lambda_2$ bo'ladi, bunda $A = 0$, $B = -2$, $C = 0$ va $B^2 - AC = 4 > 0$ ekanligidan tenglamaning berilgan $u = x$ yechimi bo'ylab tipi giperbolik tipdagi tenglama bo'ladi.

3) agar $u = x^2 + \dfrac{y^2}{4} + \dfrac{17}{16}xy$ bo'lsa, u holda $u_{xx} = 2$, $u_{xy} = \dfrac{17}{16}$, $u_{yy} = \dfrac{1}{2}$ ekanligini hisobga olsak, u holda $K = 4\lambda_1^2 - 4\lambda_1\lambda_2 + \lambda_2^2$ bo'ladi, bunda $A = 4$, $B = -2$, $C = 1$ va $B^2 - AC = 4 - 4 = 0$ ekanligidan tenglamaning berilgan $u = x^2 + \dfrac{y^2}{4} + \dfrac{17}{16}xy$ yechimi bo'ylab tipi parabolik tipdagi tenglama bo'ladi.

5-misol. Quyidagi haqiqiy o'zgaruvchili Monja–Amper tenglamasini berilgan $u(x,y)$ yechimi bo'ylab tipini aniqlang:

$$u_{xx} \cdot u_{yy} - u_{xy}^2 = 0.$$

Yechish: Bu tenglamaga mos, $F=u_{xx}\cdot u_{yy}-u_{xy}^2$, $\dfrac{\partial F}{\partial u_{xx}}=u_{yy}$, $\dfrac{\partial F}{\partial u_{xy}}=-2u_{xy}$, $\dfrac{\partial F}{\partial u_{yy}}=u_{xx}$ bo'lgani uchun xarakteristik ko'phad

$$K=\frac{\partial F}{\partial u_{xx}}\lambda_1^2+\frac{\partial F}{\partial u_{xy}}\lambda_1\lambda_2+\frac{\partial F}{\partial u_{yy}}\lambda_2^2=u_{yy}\lambda_1^2-2u_{xy}\lambda_1\lambda_2+u_{xx}\lambda_2^2$$

ko'rinishida bo'ladi. $u(x,y)$ funksiya $u_{xx}\cdot u_{yy}-u_{xy}^2=0$ tenglamaning yechimi bo'lsin. U holda $A=u_{yy}$, $B=-u_{xy}$, $C=u_{xx}$ bo'lgani uchun $B^2-AC=u_{xy}^2-u_{xx}\cdot u_{yy}=0$ bo'ladi. Shuning uchun bu tenglamaning ixtiyoriy $u(x,y)$ yechimi bo'ylab tipi parabolik tipdagi tenglama bo'ladi.

10. Xususiy hosilali differensial tenglamalar sistemasining tipini aniqlash. Bizga $u_1, u_2, \ldots, u_N$ noma'lum funksiyalar qatnashgan har biri $m-$ tartibli quyidagi N ta xususiy hosilalali differensial tenglamalar sistemasi berilgan bo'lsin:

$$\begin{cases} F_1\left(x,u_1,u_2,\ldots,u_N,\ldots,p^1_{x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}},\ldots p^N_{x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}}\right)=0 \\ F_2\left(x,u_1,u_2,\ldots,u_N,\ldots,p^1_{x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}},\ldots p^N_{x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}}\right)=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_N\left(x,u_1,u_2,\ldots,u_N,\ldots,p^1_{x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}},\ldots p^N_{x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}}\right)=0, \end{cases}$$

bu yerda $\quad p^j_{x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}}=\dfrac{\partial^{|i|}u_j}{\partial x_1^{i_1}\partial x_2^{i_2}\ldots\partial x_n^{i_n}}$, $0\le|i|\le m$, $0\le j\le N$.

Ushbu tenglamalar sistemasining tipini aniqlash uchun uning xarakteristik formasini tuzamiz. Buning uchun bizga quyidagi kvadratik matritsalar zarur bo'ladi:

$$
A_{i_1 i_2 \dots i_n} = \begin{pmatrix} \frac{\partial F_1}{\partial p^1_{i_1 i_2 \dots i_n}} & \frac{\partial F_1}{\partial p^2_{i_1 i_2 \dots i_n}} & \cdots & \frac{\partial F_1}{\partial p^N_{i_1 i_2 \dots i_n}} \\ \frac{\partial F_2}{\partial p^1_{i_1 i_2 \dots i_n}} & \frac{\partial F_2}{\partial p^2_{i_1 i_2 \dots i_n}} & \cdots & \frac{\partial F_2}{\partial p^N_{i_1 i_2 \dots i_n}} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial F_N}{\partial p^1_{i_1 i_2 \dots i_n}} & \frac{\partial F_N}{\partial p^2_{i_1 i_2 \dots i_n}} & \cdots & \frac{\partial F_N}{\partial p^N_{i_1 i_2 \dots i_n}} \end{pmatrix}, \quad \sum_{k=1}^{n} i_k = m .
$$

Bu matritsalardan foydalanib, $\lambda_1, \lambda_2, ..., \lambda_n$ haqiqiy skalyar parametrlarga nisbatan ushbu Nm – tartibli xarakteristik formani tuzamiz:

$$
K(\lambda_1, \lambda_2, \dots, \lambda_n) = \det\left(\sum_{|i|=m} A_{i_1 i_2 \dots i_n} \lambda_1^{i_1} \dots \lambda_n^{i_n} \right).
$$

Yuqoridagi sistemaning tipini aniqlash ushbu xarakteristik formaning shakliga qarab, m – tartibli bitta tenglama qaralgani singari tiplarga bo'linadi.

1–misol. $\begin{cases} 2u_x - 4v_x + 3u_y + 8v_y - u = 0 \\ 3u_x - 2v_x + 6u_y + 3v_y + 2u = 0 \end{cases}$ tenglamalar sistemasining tipini aniqlang.

Yechish: Avvalambor, biz $A_{i_1 i_2 \dots i_n}$, $\sum_{k=1}^{n} i_k = m$ matritsalarni tuzamiz. Bizning misolda N=2, n=2, $\sum_{k=1}^{2} i_k = 1$, $u_1 = u$, $u_2 = v$ bo'lgani uchun $A_{i_1 i_2 \dots i_n}$ matritsalar quyidagicha bo'ladi:

$$A_{10}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_x} & \dfrac{\partial F_1}{\partial v_x}\\ \dfrac{\partial F_2}{\partial u_x} & \dfrac{\partial F_2}{\partial v_x}\end{pmatrix}=\begin{pmatrix}2 & -4\\ 3 & -2\end{pmatrix},\ \text{bu yerda } i_1=1,\, i_2=0,$$

$$A_{01}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_y} & \dfrac{\partial F_1}{\partial v_y}\\ \dfrac{\partial F_2}{\partial u_y} & \dfrac{\partial F_2}{\partial v_y}\end{pmatrix}=\begin{pmatrix}3 & 8\\ 6 & 3\end{pmatrix},\ \text{bu yerda } i_1=0,\, i_2=1.$$

Endi esa, $K(\lambda_1,\lambda_2,\dots,\lambda_n)=\det\left(\sum_{|i|=m} A_{i_1 i_2\dots i_n}\lambda_1^{i_1}\dots\lambda_n^{i_n}\right)$ xarakteristik ko'phadni tuzamiz:

$$K(\lambda_1,\lambda_2)=\det\left(\begin{pmatrix}2 & -4\\ 3 & -2\end{pmatrix}\lambda_1+\begin{pmatrix}3 & 8\\ 6 & 3\end{pmatrix}\lambda_2\right)=$$

$$=\det\begin{pmatrix}2\lambda_1+3\lambda_2 & -4\lambda_1+8\lambda_2\\ 3\lambda_1+6\lambda_2 & -2\lambda_1+3\lambda_2\end{pmatrix}=-4\lambda_1^2+9\lambda_2^2+12\lambda_1^2-48\lambda_2^2=$$

$$=8\lambda_1^2-39\lambda_2^2.$$

$B^2-AC=0^2-8\cdot(-39)=312>0$. Demak, berilgan tenglamalar sistemasi tekislikning hamma nuqtalarida giperbolik tipda bo'ladi.

2–misol. $\begin{cases}u_x+u_y+v_y+v_z-xyu=0\\ v_x-u_y-v_y+u_z+2u=0\end{cases}$ tenglamalar sistemasining tipini aniqlang.

Yechish: Avvalambor, biz $A_{i_1 i_2\dots i_n}$, $\sum_{k=1}^{n} i_k=m$ matritsalarni tuzamiz. Bizning misolda N=2, n=3, $\sum_{k=1}^{3} i_k=1$, $u_1=u$, $u_2=v$ bo'lgani uchun $A_{i_1 i_2\dots i_n}$ matritsalar quyidagicha bo'ladi:

$$A_{100} = \begin{pmatrix} \dfrac{\partial F_1}{\partial u_x} & \dfrac{\partial F_1}{\partial v_x} \\ \dfrac{\partial F_2}{\partial u_x} & \dfrac{\partial F_2}{\partial v_x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ bu yerda } i_1 = 1,\ i_2 = 0,\ i_3 = 0,$$

$$A_{010} = \begin{pmatrix} \dfrac{\partial F_1}{\partial u_y} & \dfrac{\partial F_1}{\partial v_y} \\ \dfrac{\partial F_2}{\partial u_y} & \dfrac{\partial F_2}{\partial v_y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \text{ bu yerda } i_1 = 0,\ i_2 = 1,\ i_3 = 0,$$

$$A_{001} = \begin{pmatrix} \dfrac{\partial F_1}{\partial u_z} & \dfrac{\partial F_1}{\partial v_z} \\ \dfrac{\partial F_2}{\partial u_z} & \dfrac{\partial F_2}{\partial v_z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ bu yerda } i_1 = 0,\ i_2 = 0,\ i_3 = 1.$$

Endi esa $K(\lambda_1, \lambda_2, \ldots, \lambda_n) = \det\left(\sum_{|i|=m} A_{i_1 i_2 \ldots i_n} \lambda_1^{i_1} \ldots \lambda_n^{i_n}\right)$

xarakteristik ko'phadni tuzamiz:

$$K(\lambda_1, \lambda_2, \lambda_3) =$$

$$= \det\left(\begin{pmatrix} \dfrac{\partial F_1}{\partial u_x} & \dfrac{\partial F_1}{\partial v_x} \\ \dfrac{\partial F_2}{\partial u_x} & \dfrac{\partial F_2}{\partial v_x} \end{pmatrix} \lambda_1 + \begin{pmatrix} \dfrac{\partial F_1}{\partial u_y} & \dfrac{\partial F_1}{\partial v_y} \\ \dfrac{\partial F_2}{\partial u_y} & \dfrac{\partial F_2}{\partial v_y} \end{pmatrix} \lambda_2 + \begin{pmatrix} \dfrac{\partial F_1}{\partial u_z} & \dfrac{\partial F_1}{\partial v_z} \\ \dfrac{\partial F_2}{\partial u_z} & \dfrac{\partial F_2}{\partial v_z} \end{pmatrix} \lambda_3 \right) =$$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \lambda_1 + \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \lambda_2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \lambda_3 \right) =$$

$$= \det\begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_2 + \lambda_3 \\ -\lambda_2 + \lambda_3 & \lambda_1 - \lambda_2 \end{pmatrix} = \lambda_1^2 - \lambda_2^2 + \lambda_2^2 - \lambda_3^2 = \lambda_1^2 - \lambda_3^2.$$

Xarakteristik formaning $K(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 - \lambda_3^2$ kanonik shaklida ikkinchi koeffitsient nolga tengdir. Shunga ko'ra, berilgan tenglamalar sistemasi fazoning hamma nuqtalarida parabolik tipda bo'ladi.

3-misol. $\begin{cases} u_x + u_y + v_y - u = 0 \\ v_x - 2u_y - v_y + xu = 0 \end{cases}$ tenglamalar sistemasining tipini aniqlang.

Yechish: Avvalambor, biz $A_{i_1 i_2 \ldots i_n}$, $\sum_{k=1}^{n} i_k = m$ matritsalarni tuzamiz. Bizning misolda N=2, n=2, $\sum_{k=1}^{2} i_k = 1$, $u_1 = u$, $u_2 = v$ bo'lgani uchun $A_{i_1 i_2 \ldots i_n}$ matritsalar quyidagicha bo'ladi:

$$A_{10} = \begin{pmatrix} \dfrac{\partial F_1}{\partial u_x} & \dfrac{\partial F_1}{\partial v_x} \\ \dfrac{\partial F_2}{\partial u_x} & \dfrac{\partial F_2}{\partial v_x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ bu yerda } i_1 = 1,\ i_2 = 0,$$

$$A_{01} = \begin{pmatrix} \dfrac{\partial F_1}{\partial u_y} & \dfrac{\partial F_1}{\partial v_y} \\ \dfrac{\partial F_2}{\partial u_y} & \dfrac{\partial F_2}{\partial v_y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \text{ bu yerda } i_1 = 0,\ i_2 = 1.$$

Endi esa $K(\lambda_1, \lambda_2, \ldots, \lambda_n) = \det\left(\sum_{|i|=m} A_{i_1 i_2 \ldots i_n} \lambda_1^{i_1} \ldots \lambda_n^{i_n} \right)$

xarakteristik ko'phadni tuzamiz:

$$K(\lambda_1, \lambda_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \lambda_1 + \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \lambda_2 \right) =$$

$$= \det\begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_2 \\ -2\lambda_2 & \lambda_1 - \lambda_2 \end{pmatrix} = \lambda_1^2 - \lambda_2^2 + 2\lambda_2^2 = \lambda_1^2 + \lambda_2^2.$$

Xarakteristik formaning $K(\lambda_1, \lambda_2) = \lambda_1^2 + \lambda_2^2$ kanonik shaklida ikkala koeffitsient ham birga tengdir. Shunga ko'ra, berilgan tenglamalar sistemasi tekislikning hamma nuqtalarida elliptik tipda bo'ladi.

Agar berilgan tenglama yoki sistemadagi erkli o'zgaruvchilar $z_k = x_k + iy_k$, $k = 1,2,...,n$ kompleks o'zgaruvchilar bo'lsa, u holda hosilalarni

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right), \quad \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right)$$

deb tushunib, berilgan tenglama yoki sistemani unga teng kuchli bo'lgan tenglama yoki sistemaga keltiramiz. $D \subset C^n$ sohaning har bir $z \in D$ nuqtasida $f = u + iv$ funksiya haqiqiy analiz ma'nosida differensiallanuvchi, ya'ni

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial y_1}dy_1 + ... + \frac{\partial f}{\partial x_n}dx_n + \frac{\partial f}{\partial y_n}dy_n$$

differensial mavjud bo'lsin. U holda

$$df = \frac{\partial f}{\partial z_1}dz_1 + \frac{\partial f}{\partial \bar{z}_1}d\bar{z}_1 + ... + \frac{\partial f}{\partial z_n}dz_n + \frac{\partial f}{\partial \bar{z}_n}d\bar{z}_n$$

differensial ham mavjud bo'ladi, ya'ni

$$df = \sum_{k=1}^{n}\left(\frac{\partial f}{\partial z_k}dz_k + \frac{\partial f}{\partial \bar{z}_k}d\bar{z}_k\right).$$

Bu yerda

$$\frac{\partial f}{\partial z_k} = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} - i\frac{\partial f}{\partial y_k}\right), \quad \frac{\partial f}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} + i\frac{\partial f}{\partial y_k}\right)$$

belgilashlar kiritilgan.

Shuningdek,

$$\partial = \sum_{k=1}^{n}\frac{\partial}{\partial z_k}dz_k, \quad \bar{\partial} = \sum_{k=1}^{n}\frac{\partial}{\partial \bar{z}_k}d\bar{z}_k, \quad d = \partial + \bar{\partial}$$

belgilashlarni ham kiritsak, u holda $z \in C^n$ nuqtada $f = u + iv$ funksiya $\mathbb{R}$ – differensiallanuvchi (haqiqiy analiz ma'nosida) bo'lib, uning $\mathbb{C}$ – differensiallanuvchi bo'lishligi uchun

$$\bar{\partial} f = \sum_{k=1}^{n}\frac{\partial f}{\partial \bar{z}_k}d\bar{z}_k = 0$$

Koshi–Riman shartining bajarilishi zarur va yetarli bo'ladi. Koshi–Riman sharti n ta

$$\frac{\partial f}{\partial \bar{z}_k}=\frac{1}{2}\left(\frac{\partial f}{\partial x_k}+i\frac{\partial f}{\partial y_k}\right)=0,\quad k=1,...,n$$

kompleks tenglamalar sistemasiga yoki $2n$ ta

$$\frac{\partial u}{\partial x_k}=\frac{\partial v}{\partial y_k},\quad \frac{\partial u}{\partial y_k}=-\frac{\partial v}{\partial x_k},\quad k=1,...,n$$

haqiqiy tenglamalar sistemasiga ekvivalent bo'ladi, bunda $u=\operatorname{Re} f$, $v=\operatorname{Im} f$. Bu tenglamalar sistemasi ikkita noma'lum funksiyaga nisbatan $2n$ ta tenglamani saqlaydi. Shuning uchun $n>1$ bo'lgan ko'p o'lchamli kompleks analiz bir o'lchamli kompleks analizdan farq qiladi.

Agar $f=u+iv$ funksiya $z\in C^n$ nuqtaning qandaydir atrofida $\mathbb{C}$ – differensiallanuvchi bo'lsa, u holda bu $f=u+iv$ funksiya $z\in C^n$ nuqtada golomorf funksiya deb ataladi. Ochiq to'plamda $\mathbb{C}$ – differensiallanuvchi bo'lishlik va golomorf funksiya bo'lishlik tushunchalari ustma–ust tushadi. Agar $f=u+iv$ funksiya $z\in C^n$ nuqtada golomorf funksiya bo'lsa, u holda $\bar{f}=u-iv$ funksiya bu nuqta atrofida $\mathbb{R}$ – differensiallanuvchi bo'lib, har bir $k=1,2,...,n$ uchun

$$\frac{\partial \bar{f}}{\partial z_k}=\frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x_k}-i\frac{\partial \bar{f}}{\partial y_k}\right)=\overline{\left(\frac{\partial f}{\partial \bar{z}_k}\right)}=0$$

tengliklar o'rinli bo'ladi. Bunday $\bar{f}=u-iv$ funksiya bu $z\in C^n$ nuqtada *antigolomorf funksiya* deb ataladi.

$f=u+iv$ funksiya $z\in C^n$ nuqtada golomorf funksiya bo'lsin. U holda bu funksiyaning $u=\frac{1}{2}(f+\bar{f})$ haqiqiy qismi shu $z\in C^n$ nuqta atrofida $\frac{\partial u}{\partial z_k}=\frac{1}{2}\frac{\partial f}{\partial z_k}$ hosilaga ega bo'ladi. Golomorf funksiyaning xususiy hosilasi ham golomorf funksiya ekanligidan har bir $k=1,2,...,n$ va har bir $m=1,2,...,n$ uchun

$$\frac{\partial^2 u}{\partial z_k \partial \bar{z}_m} = \frac{\partial}{\partial \bar{z}_m}\left(\frac{\partial u}{\partial z_k}\right) = 0$$

tengliklar o'rinli bo'ladi. Bu tengliklarning haqiqiy va mavhum qismlarini ajratsak, u holda

$$\frac{\partial}{\partial \bar{z}_m}\frac{\partial}{\partial z_k} = \frac{1}{4}\left(\frac{\partial^2}{\partial x_m \partial x_k} + \frac{\partial^2}{\partial y_m \partial y_k}\right) + \frac{i}{4}\left(\frac{\partial^2}{\partial y_m \partial x_k} - \frac{\partial^2}{\partial x_m \partial y_k}\right) = 0,$$

tenglik o'rinli bo'lib, n^2 ta ikkinchi tartibli

$$\frac{\partial^2 u}{\partial x_m \partial x_k} + \frac{\partial^2 u}{\partial y_m \partial y_k} = 0, \qquad \frac{\partial^2 u}{\partial y_m \partial x_k} - \frac{\partial^2 u}{\partial x_m \partial y_k} = 0$$

xususiy hosilali differensial tenglamalarga ajraladi. Agar ∂ va $\bar{\partial}$ operatorlardan foydalansak, u holda har bir $k = 1,2,...,n$ va har bir $m = 1,2,...,n$ uchun

$$\frac{\partial^2 u}{\partial z_k \partial \bar{z}_m} = \frac{\partial}{\partial \bar{z}_m}\left(\frac{\partial u}{\partial z_k}\right) = 0$$

tengliklarni bitta

$$\partial\bar{\partial}u = 0$$

shart bilan yozish mumkin.

$D \subset C^n$ sohada C^2 sinfga tegishli bo'lgan u funksiya har bir $z \in D$ nuqtada $\partial\bar{\partial}u = 0$ shartni qanoatlantirsa, u holda bu u funksiya D sohada *plyurigarmonik funksiya* deb ataladi.

$u \in C^2(D)$ funksiya D sohada plyurigarmonik funksiya bo'lishligi uchun uning ixtiyoriy $\{z = z^0 + \omega\zeta\}$ kompleks chiziqdagi qisqartmasi, ya'ni $h(\zeta) = u(z^0 + \omega\zeta)$ funksiyaning $\{\zeta \in \mathbb{C} : z^0 + \omega\zeta \in D\}$ ochiq to'plamda garmonik funksiya bo'lishligi zarur va yetarli bo'ladi.

Agar $u \in C^2(D)$ funksiyaning ixtiyoriy ikki o'lchamli $\{z = z^0 + \omega\zeta + \omega'\zeta\}$ kompleks tekislikka qisqartmasi garmonik funksiya bo'lsa, u holda u – chiziqli funksiya bo'ladi.

$D \subset C^n$ sohada golomorf bo'lgan $f = u + iv$ funksiyaning u haqiqiy va v mavhum qismlari shu sohada plyurigarmonik funksiya bo'ladi. Bu tasdiqning teskarisi umuman olganda, lokal holda o'rinli bo'ladi, ya'ni $(x^0, y^0) \in R^{2n}$ nuqtaning U atrofida plyurigarmonik bo'lgan ixtiyoriy u funksiya uchun haqiqiy (yoki mavhum) qismi u funksiya bo'lgan va $z^0 = x^0 + iy^0$ nuqtada golomorf bo'lgan $f = u + iv$ funksiya mavjud.

Har bir $m = k$, $k = 1, 2, ..., n$ uchun

$$\frac{\partial^2 u}{\partial x_k^2} + \frac{\partial^2 u}{\partial y_k^2} = 0$$

tengliklarni qo'shib,

$$\Delta u = \sum_{k=1}^{n} \left(\frac{\partial^2 u}{\partial x_k^2} + \frac{\partial^2 u}{\partial y_k^2} \right) = 0$$

Laplas tenglamasiga ega bo'lamiz. Shunga ko'ra, $n > 1$ uchun $\mathbb{R}^{2n}$ fazodagi plyurigarmonik funksiyalar sinfi garmonik funksiyalar sinfining qismini tashkil etadi.

$D \subset C^n$ sohaning har bir $z \in D$ nuqtasida

$$\frac{\partial f}{\partial \overline{z}_1} = 0,\ \frac{\partial f}{\partial \overline{z}_2} = 0,\ ...,\ \frac{\partial f}{\partial \overline{z}_n} = 0$$

tenglamalar sistemasi o'rinli bo'lsin, ya'ni

$$\frac{\partial f}{\partial \overline{z}_1} = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial y_1} \right) = 0,\ ...,\ \frac{\partial f}{\partial \overline{z}_n} = \frac{1}{2}\left(\frac{\partial f}{\partial x_n} + i\frac{\partial f}{\partial y_n} \right) = 0$$

tenglamalar sistemasi o'rinli bo'lsin. U holda

$$\frac{\partial u}{\partial x_1} - \frac{\partial v}{\partial y_1} = 0,\ \ \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial y_1} = 0,\ ...,\ \frac{\partial u}{\partial x_n} - \frac{\partial v}{\partial y_n} = 0,\ \frac{\partial v}{\partial x_n} + \frac{\partial u}{\partial y_n} = 0$$

tenglamalar sistemasi o'rinli bo'ladi. Bu sistemaning $2n$ – tartibli xarakteristik formasini tuzamiz.

Buning uchun bizga quyidagi kvadratik matritsalar zarur bo'ladi:

$$A_{10...00}=\begin{pmatrix}1 & 0 & \ldots & 0 & 0\\ 0 & 1 & \ldots & 0 & 0\\ . & . & \ldots & . & .\\ 0 & 0 & \ldots & 0 & 0\\ 0 & 0 & \ldots & 0 & 0\end{pmatrix},\quad A_{01...00}=\begin{pmatrix}0 & -1 & \ldots & 0 & 0\\ 1 & 0 & \ldots & 0 & 0\\ . & . & \ldots & . & .\\ 0 & 0 & \ldots & 0 & 0\\ 0 & 0 & \ldots & 0 & 0\end{pmatrix},\ldots,$$

$$A_{00...10}=\begin{pmatrix}0 & 0 & \ldots & 0 & 0\\ 0 & 0 & \ldots & 0 & 0\\ . & . & \ldots & . & .\\ 0 & 0 & \ldots & 1 & 0\\ 0 & 0 & \ldots & 0 & 1\end{pmatrix},\quad A_{00...01}=\begin{pmatrix}0 & 0 & \ldots & 0 & 0\\ 0 & 0 & \ldots & 0 & 0\\ . & . & \ldots & . & .\\ 0 & 0 & \ldots & 0 & -1\\ 0 & 0 & \ldots & 1 & 0\end{pmatrix}$$

Bu matritsalardan foydalanib, $\lambda_1, \lambda_2, ..., \lambda_{2n-1}, \lambda_{2n}$ haqiqiy skalyar parametrlarga nisbatan ushbu $2n$ – tartibli xarakteristik formani tuzamiz:

$$K(\lambda_1,\lambda_2,\ldots,\lambda_{2n-1},\lambda_{2n})=$$

$$=\det\left(\sum_{|i|=1} A_{i_1 i_2 \ldots i_{2n-1}, i_{2n}} \lambda_1^{i_1}\lambda_2^{i_2}\ldots\lambda_{2n-1}^{i_{2n-1}}\lambda_{2n}^{i_{2n}}\right)=$$

$$=\det\begin{pmatrix}\lambda_1 & -\lambda_2 & \ldots & 0 & 0\\ \lambda_2 & \lambda_1 & \ldots & 0 & 0\\ . & . & \ldots & & \\ 0 & 0 & \ldots & \lambda_{2n-1} & -\lambda_{2n}\\ 0 & 0 & \ldots & \lambda_{2n} & \lambda_{2n-1}\end{pmatrix}=$$

$$=\left(\lambda_1^2+\lambda_2^2\right)\cdot\ldots\cdot\left(\lambda_{2n-1}^2+\lambda_{2n}^2\right).$$

Bu yerdan, ko'rinadiki, Koshi–Riman tenglamalari sistemasi $n=1$ uchun $\mathbb{R}^2$ fazoda elliptik tipga tegishli va $n>1$ uchun $\mathbb{R}^{2n}$ fazoda elliptik tipda bo'lmagan tenglamalar sistemasini tashkil etadi.

4-misol. Kompleks o'zgaruvchili $\begin{vmatrix} f_{z_1\bar{z}_1} & f_{z_1\bar{z}_2} \\ f_{z_2\bar{z}_1} & f_{z_2\bar{z}_2} \end{vmatrix} = 0$ Monja–Amper tenglamasini $n=2$ bo'lgan holda qaraymiz. Bu yerda $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$,

$$f = u + iv, \quad \frac{\partial f}{\partial z_1} = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - i\frac{\partial f}{\partial y_1}\right), \quad \frac{\partial f}{\partial \bar{z}_1} = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial y_1}\right),$$

$$\frac{\partial f}{\partial z_2} = \frac{1}{2}\left(\frac{\partial f}{\partial x_2} - i\frac{\partial f}{\partial y_2}\right), \quad \frac{\partial f}{\partial \bar{z}_2} = \frac{1}{2}\left(\frac{\partial f}{\partial x_2} + i\frac{\partial f}{\partial y_2}\right),$$

$$\frac{\partial^2 f}{\partial z_1 \partial \bar{z}_1} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial y_1^2}\right), \quad \frac{\partial^2 f}{\partial z_2 \partial \bar{z}_2} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial y_2^2}\right),$$

$$\frac{\partial^2 f}{\partial z_1 \partial \bar{z}_2} = \frac{1}{4}\left[\left(\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\partial^2 f}{\partial y_1 \partial y_2}\right) + i\left(\frac{\partial^2 f}{\partial x_1 \partial y_2} - \frac{\partial^2 f}{\partial x_2 \partial y_1}\right)\right],$$

$$\frac{\partial^2 f}{\partial \bar{z}_1 \partial z_2} = \frac{1}{4}\left[\left(\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\partial^2 f}{\partial y_1 \partial y_2}\right) - i\left(\frac{\partial^2 f}{\partial x_1 \partial y_2} - \frac{\partial^2 f}{\partial x_2 \partial y_1}\right)\right]$$

tengliklar o'rinli bo'ladi. Shuning uchun Monja–Amper tenglamasi

$$\frac{1}{16}\left[\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial y_1^2}\right)\left(\frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial y_2^2}\right) - \right.$$

$$\left. -\left(\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\partial^2 f}{\partial y_1 \partial y_2}\right)^2 - \left(\frac{\partial^2 f}{\partial x_1 \partial y_2} - \frac{\partial^2 f}{\partial x_2 \partial y_1}\right)^2\right] = 0$$

yoki

$$F \equiv (u+iv)_{x_1x_1} \cdot (u+iv)_{x_2x_2} + (u+iv)_{x_1x_1} \cdot (u+iv)_{y_2y_2} +$$

$$+(u+iv)_{x_2x_2} \cdot (u+iv)_{y_1y_1} + (u+iv)_{y_1y_1} \cdot (u+iv)_{y_2y_2} -$$

$$-\left((u+iv)_{x_1x_2}\right)^2 - 2\cdot(u+iv)_{x_1x_2} \cdot (u+iv)_{y_1y_2} - \left((u+iv)_{y_1y_2}\right)^2 -$$

$$-\left((u+iv)_{x_1y_2}\right)^2+2\cdot(u+iv)_{x_1y_2}\cdot(u+iv)_{x_2y_1}-\left((u+iv)_{x_2y_1}\right)^2=0$$

kompleks tenglama ko'rinishida bo'ladi. Bu tenglamaning haqiqiy va mavhum qismlarini ajratsak, u holda

$$\begin{gathered}
u_{x_1x_1}\cdot u_{x_2x_2}-v_{x_1x_1}\cdot v_{x_2x_2}+u_{x_1x_1}\cdot u_{y_2y_2}-v_{x_1x_1}v_{y_2y_2}+\\
+u_{x_2x_2}\cdot u_{y_1y_1}-v_{x_2x_2}\cdot v_{y_1y_1}+u_{y_1y_1}\cdot u_{y_2y_2}-v_{y_1y_1}\cdot v_{y_2y_2}+\\
-\left(u_{x_1x_2}\right)^2+\left(v_{x_1x_2}\right)^2-2\cdot u_{x_1x_2}\cdot u_{y_1y_2}+2\cdot v_{x_1x_2}\cdot v_{y_1y_2}-\\
-\left(u_{y_1y_2}\right)^2+\left(v_{y_1y_2}\right)^2-\left(u_{x_1y_2}\right)^2+\left(v_{x_1y_2}\right)^2+\\
+2\cdot u_{x_1y_2}\cdot u_{x_2y_1}-2\cdot v_{x_1y_2}\cdot v_{x_2y_1}-\left(u_{x_2y_1}\right)^2+\left(v_{x_2y_1}\right)^2=0,\\
u_{x_1x_1}\cdot v_{x_2x_2}+v_{x_1x_1}\cdot u_{x_2x_2}+u_{x_1x_1}\cdot v_{y_2y_2}+v_{x_1x_1}\cdot u_{y_2y_2}+\\
+u_{x_2x_2}\cdot v_{y_1y_1}+v_{x_2x_2}\cdot u_{y_1y_1}+u_{y_1y_1}\cdot v_{y_2y_2}+v_{y_1y_1}\cdot u_{y_2y_2}-\\
-2u_{x_1x_2}v_{x_1x_2}-2\cdot u_{x_1x_2}\cdot v_{y_1y_2}-2\cdot v_{x_1x_2}\cdot u_{y_1y_2}-2u_{y_1y_2}v_{y_1y_2}-\\
-2u_{x_1y_2}v_{x_1y_2}+2\cdot u_{x_1y_2}\cdot v_{x_2y_1}+2\cdot v_{x_1y_2}\cdot u_{x_2y_1}-2u_{x_2y_1}v_{x_2y_1}=0
\end{gathered}$$

haqiqiy tenglamalar sistemasiga ega bo'lamiz. Bu tenglamalar sistemasiga mos bo'lgan 4 – tartibli xarakteristik formani tuzamiz. Avvalambor, biz $A_{i_1i_2\ldots i_n}$, $\sum_{k=1}^{n} i_k=m$ matritsalarni tuzamiz. Bizning misolda $N=2$, $n=4$,

$\sum_{k=1}^{4} i_k=2$, $u_1=u$, $u_2=v$ bo'lgani uchun $A_{i_1i_2i_3i_4}=\begin{pmatrix}\dfrac{\partial F_1}{\partial p^1_{i_1i_2i_3i_4}} & \dfrac{\partial F_1}{\partial p^2_{i_1i_2i_3i_4}}\\ \dfrac{\partial F_2}{\partial p^1_{i_1i_2i_3i_4}} & \dfrac{\partial F_2}{\partial p^2_{i_1i_2i_3i_4}}\end{pmatrix}$

matritsalar quyidagicha bo'ladi:

$$A_{2000}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_{x_1x_1}} & \dfrac{\partial F_1}{\partial v_{x_1x_1}}\\ \dfrac{\partial F_2}{\partial u_{x_1x_1}} & \dfrac{\partial F_2}{\partial v_{x_1x_1}}\end{pmatrix}=\begin{pmatrix}u_{x_2x_2}+u_{y_2y_2} & -v_{x_2x_2}-v_{y_2y_2}\\ v_{x_2x_2}+v_{y_2y_2} & u_{x_2x_2}+u_{y_2y_2}\end{pmatrix},$$

$$A_{1100}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_{x_1y_1}} & \dfrac{\partial F_1}{\partial v_{x_1y_1}}\\[2mm] \dfrac{\partial F_2}{\partial u_{x_1y_1}} & \dfrac{\partial F_2}{\partial v_{x_1y_1}}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},$$

$$A_{1010}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_{x_1x_2}} & \dfrac{\partial F_1}{\partial v_{x_1x_2}}\\[2mm] \dfrac{\partial F_2}{\partial u_{x_1x_2}} & \dfrac{\partial F_2}{\partial v_{x_1x_2}}\end{pmatrix}=\begin{pmatrix}-2u_{x_1x_2}-2u_{y_1y_2} & 2v_{x_1x_2}+2v_{y_1y_2}\\ -2v_{x_1x_2}-2v_{y_1y_2} & -2u_{x_1x_2}-2u_{y_1y_2}\end{pmatrix},$$

$$A_{1001}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_{x_1y_2}} & \dfrac{\partial F_1}{\partial v_{x_1y_2}}\\[2mm] \dfrac{\partial F_2}{\partial u_{x_1y_2}} & \dfrac{\partial F_2}{\partial v_{x_1y_2}}\end{pmatrix}=\begin{pmatrix}-2u_{x_1y_2}+2u_{x_2y_1} & 2v_{x_1y_2}-2v_{x_2y_1}\\ -2v_{x_1y_2}+2v_{x_2y_1} & -2u_{x_1y_2}+2u_{x_2y_1}\end{pmatrix},$$

$$A_{0200}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_{y_1y_1}} & \dfrac{\partial F_1}{\partial v_{y_1y_1}}\\[2mm] \dfrac{\partial F_2}{\partial u_{y_1y_1}} & \dfrac{\partial F_2}{\partial v_{y_1y_1}}\end{pmatrix}=\begin{pmatrix}u_{x_2x_2}+u_{y_2y_2} & -v_{x_2x_2}-v_{y_2y_2}\\ v_{x_2x_2}+v_{y_2y_2} & u_{x_2x_2}+u_{y_2y_2}\end{pmatrix},$$

$$A_{0110}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_{y_1x_2}} & \dfrac{\partial F_1}{\partial v_{y_1x_2}}\\[2mm] \dfrac{\partial F_2}{\partial u_{y_1x_2}} & \dfrac{\partial F_2}{\partial v_{y_1x_2}}\end{pmatrix}=\begin{pmatrix}2u_{x_1y_2}-2u_{y_1x_2} & -2v_{x_1y_2}+2v_{y_1x_2}\\ 2v_{x_1y_2}-2v_{y_1x_2} & 2u_{x_1y_2}-2u_{y_1x_2}\end{pmatrix},$$

$$A_{0101}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_{y_1y_2}} & \dfrac{\partial F_1}{\partial v_{y_1y_2}}\\[2mm] \dfrac{\partial F_2}{\partial u_{y_1y_2}} & \dfrac{\partial F_2}{\partial v_{y_1y_2}}\end{pmatrix}=\begin{pmatrix}-2u_{x_1x_2}-2u_{y_1y_2} & 2v_{x_1x_2}+2v_{y_1y_2}\\ -2v_{x_1x_2}-2v_{y_1y_2} & -2u_{x_1x_2}-2u_{y_1y_2}\end{pmatrix},$$

$$A_{0020}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_{x_2x_2}} & \dfrac{\partial F_1}{\partial v_{x_2x_2}}\\[2mm] \dfrac{\partial F_2}{\partial u_{x_2x_2}} & \dfrac{\partial F_2}{\partial v_{x_2x_2}}\end{pmatrix}=\begin{pmatrix}u_{x_1x_1}+u_{y_1y_1} & -v_{x_1x_1}-v_{y_1y_1}\\ v_{x_1x_1}+v_{y_1y_1} & u_{x_1x_1}+u_{y_1y_1}\end{pmatrix},$$

$$A_{0011}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_{x_2y_2}} & \dfrac{\partial F_1}{\partial v_{x_2y_2}}\\ \dfrac{\partial F_2}{\partial u_{x_2y_2}} & \dfrac{\partial F_2}{\partial v_{x_2y_2}}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},$$

$$A_{0002}=\begin{pmatrix}\dfrac{\partial F_1}{\partial u_{y_2y_2}} & \dfrac{\partial F_1}{\partial v_{y_2y_2}}\\ \dfrac{\partial F_2}{\partial u_{y_2y_2}} & \dfrac{\partial F_2}{\partial v_{y_2y_2}}\end{pmatrix}=\begin{pmatrix}u_{x_1x_1}+u_{y_1y_1} & -v_{x_1x_1}-v_{y_1y_1}\\ v_{x_1x_1}+v_{y_1y_1} & u_{x_1x_1}+u_{y_1y_1}\end{pmatrix}.$$

Endi esa, $K(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\det\left(\sum_{|i|=2}A_{i_1i_2i_3i_4}\lambda_1^{i_1}\lambda_2^{i_2}\lambda_3^{i_3}\lambda_4^{i_4}\right)$

xarakteristik ko‘phadni tuzamiz. Shunga ko‘ra,

$$K(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\det\left(\sum_{|i|=2}A_{i_1i_2i_3i_4}\lambda_1^{i_1}\lambda_2^{i_2}\lambda_3^{i_3}\lambda_4^{i_4}\right)=$$

$$=\det\left(\begin{pmatrix}u_{x_2x_2}+u_{y_2y_2} & -v_{x_2x_2}-v_{y_2y_2}\\ v_{x_2x_2}+v_{y_2y_2} & u_{x_2x_2}+u_{y_2y_2}\end{pmatrix}\left(\lambda_1^2+\lambda_2^2\right)+\right.$$

$$+\begin{pmatrix}-2u_{x_1x_2}-2u_{y_1y_2} & 2v_{x_1x_2}+2v_{y_1y_2}\\ -2v_{x_1x_2}-2v_{y_1y_2} & -2u_{x_1x_2}-2u_{y_1y_2}\end{pmatrix}(\lambda_1\lambda_3+\lambda_2\lambda_4)+$$

$$+\begin{pmatrix}-2u_{x_1y_2}+2u_{x_2y_1} & 2v_{x_1y_2}-2v_{x_2y_1}\\ -2v_{x_1y_2}+2v_{x_2y_1} & -2u_{x_1y_2}+2u_{x_2y_1}\end{pmatrix}(\lambda_1\lambda_4-\lambda_2\lambda_3)+$$

$$\left.+\begin{pmatrix}u_{x_1x_1}+u_{y_1y_1} & -v_{x_1x_1}-v_{y_1y_1}\\ v_{x_1x_1}+v_{y_1y_1} & u_{x_1x_1}+u_{y_1y_1}\end{pmatrix}\left(\lambda_3^2+\lambda_4^2\right)\right)$$

ko‘rinishda bo‘ladi. Berilgan tenglamalar sistemasining $u(x_1,y_1,x_2,y_2)$, $v(x_1,y_1,x_2,y_2)$ xususiy yechimi ma'lum bo‘lsa, u holda bu sistemani shu yechim bo‘ylab, tiplarga ajratish mumkin bo‘ladi.

Xususiy hosilali yuqori tartibli tenglamalar va sistemalarni klassifikatsiyalash bo‘yicha asosiy natijalar I.G. Petrovskiyga tegishli.

Xususiy hosilali tenglamalar nazariyasi uchun klassifikatsiyalash katta ahamiyatga ega, chunki tenglamaning u yoki boshqa tipga tegishli bo'lishligi bu tenglama yechimining ko'pgina xossalari haqida gapirish va bu tenglama uchun chegaraviy masalani qo'yishda muhim bo'ladi.

Ta'kidlash kerakki, xususiy hosilali yuqori tartibli tenglamalar va sistemalarni klassifikatsiyalashda uchta muhim: elliptik, giperbolik va parabolik tiplar ajratiladi. Shu bilan birga, xususiy hosilali yuqori tartibli tenglamalar va sistemalarning ko'p qismi bu uchta tiplarning hech biriga tegishli bo'lmaydi. Masalan, gidrodinamika sistemalari bunga misol bo'la oladi.

Hozirgi davrda ham, klassifikatsiyalash davom etmoqda. Masalan, gipoelliptik tenglamalarni o'rganish xarakteristik tenglama ildizlarining xossalari bilan emas, balki, shu tenglama yechimining xossalari bilan, aynan, uning silliqligi bilan xarakterlanadi[1].

Yuqorida aytilganidek, agar har qanday $|\xi| \neq 0$ uchun $\sum\limits_{|\alpha|=m} a_\alpha(x^0)\xi^\alpha \neq 0$ (boshqacha qilib aytganda, uning haqiqiy xarakteristikasi yo'q) bo'lsa, u holda $\sum\limits_{|\alpha|\le m} a_\alpha(x)D^\alpha u = f(x)$ chiziqli tenglama x^0 nuqtada elliptik tipdagi tenglama deyiladi. Agar Ω sohaning har bir $x^0 \in \Omega$ nuqtasida tenglama elliptik tipda bo'lsa, u holda bu tenglama Ω sohada elliptik tipdagi tenglama deyiladi. Agar elliptik tipdagi tenglamaning koeffitsientlari haqiqiy qiymatli funksiyalar bo'lsa, u holda m tenglama tartibi $m = 2l$ juft sondan iborat ekanligi algebraik teoremalardan natija sifatida kelib chiqadi.

Shuningdek, agar $\sum\limits_{j=1}^{n-1} \xi_j^2 \neq 0$ bo'ladigan har qanday $\xi_1, \dots, \xi_{n-1}$ uchun $\sum\limits_{|\alpha|=m} a_\alpha(x^0)\xi_1^{\alpha_1} \dots \xi_n^{\alpha_n} = 0$ tenglama ξ_n o'zgaruvchiga nisbatan faqat haqiqiy ildizlarga ega bo'lsa, u holda

---

[1] Bu haqida Хёрмандер Л. Линейные дифференциальные операторы с частными производными. М.: Мир, 1965. 380 стр., kitobi orqali tanishish mumkin.

$\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha u = f(x)$ chiziqli tenglama x^0 nuqtada x_n o'q yo'nalishida giperbolik tipdagi tenglama deyiladi. Agar $\sum_{j=1}^{n-1}\xi_j^2 \neq 0$ bo'ladigan har qanday $\xi_1,...,\xi_{n-1}$ uchun $\sum_{|\alpha|=m} a_\alpha(x^0)\xi_1^{\alpha_1}...\xi_n^{\alpha_n} = 0$ tenglama ξ_n o'zgaruvchiga nisbatan m ta haqiqiy va har xil ildizlarga ega bo'lsa, u holda bu $\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha u = f(x)$ chiziqli tenglama x^0 nuqtada x_n o'q yo'nalishida qa'tiy giperbolik tipdagi tenglama yoki I.G. Petrovskiy bo'yicha giperbolik tipdagi tenglama deyiladi. Agar $\sum_{j=1}^{n-1}\xi_j^2 \neq 0$ bo'ladigan har qanday $\xi_1,...,\xi_{n-1}$ uchun $\sum_{|\alpha|=m} a_\alpha(x^0)\xi_1^{\alpha_1}...\xi_n^{\alpha_n} = 0$ tenglamaning ξ_n o'zgaruvchiga nisbatan m ta haqiqiy ildizlari orasida karralilari bo'lsa, u holda bu $\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha u = f(x)$ chiziqli tenglama x^0 nuqtada x_n o'q yo'nalishida sust giperbolik tipdagi tenglama deyiladi.

Agar Ω sohaning har bir $x^0 \in \Omega$ nuqtasida tenglama giperbolik tipda bo'lsa, u holda bu tenglama Ω sohada giperbolik tipdagi tenglama deyiladi.

I.G. Petrovskiy tomonidan, m – tartibli tenglamalarning yana bir muhim sinfi ajratilgan bo'lib, Petrovskiy bo'yicha parabolik tipdagi tenglama nomiga ega.

Bu sinfni aniqlashda tenglama hadlarining $u(x)$ funksiyadan olingan nafaqat m – tartibli hosilalari, balki hosilalarining kichik tartibli hadlari ham hisobga olinadi. Ushbu

$$\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha u = f(x)$$

chiziqli tenglamani

$$\sum_{|\alpha|\le m} a_\alpha(x) D_{x'}^{\alpha'} D_t^{\alpha_n} u = f(x)$$

ko'rinishida yozamiz, bunda $x_n = t$, $x' = (x_1, ..., x_{n-1})$, $\alpha = (\alpha', \alpha_n)$, $\alpha' = (\alpha_1, ..., \alpha_{n-1})$ deb belgilaymiz.

p – butun musbat son bo'lsin. Ushbu

$$\sum_{|\alpha'|+p\alpha_n=m} a_\alpha(x)(\xi')^{\alpha'} \xi_n^{\alpha_n}$$

ko'rinishidagi ko'phadga $\sum_{|\alpha|\le m} a_\alpha(x) D_{x'}^{\alpha'} D_t^{\alpha_n} u = f(x)$ tenglama uchun x nuqtadagi p *zichlik bilan umumlashgan xarakteristik forma* deb aytiladi.

Agar qandaydir p uchun va $|\xi'|^2 = \sum_{j=1}^{n-1} \xi_j^2 = 1$ bo'ladigan ixtiyoriy haqiqiy $\xi' = (\xi_1, ..., \xi_{n-1})$ uchun

$$\sum_{|\alpha'|+p\alpha_n=m} a_\alpha(x)(i\xi')^{\alpha'} \xi_n^{\alpha_n} = 0$$

ko'rinishidagi tenglamaning ξ_n o'zgaruvchiga nisbatan barcha ildizlarining haqiqiy qismi

$$\operatorname{Re}\xi_n \le -\delta,$$

bunda $\delta = const > 0$ bo'lgan holda tengsizlikni qanoatlantirsa, u holda $\sum_{|\alpha|\le m} a_\alpha(x) D_{x'}^{\alpha'} D_t^{\alpha_n} u = f(x)$ tenglama $x = (x_1, ..., x_{n-1}, t)$ nuqtada Petrovskiy bo'yicha parabolik tipdagi tenglama deb ataladi.

Agar Ω sohaning har bir $x \in \Omega$ nuqtasida tenglama Petrovskiy bo'yicha parabolik tipdagi tenglama bo'lsa, u holda bu tenglama Ω sohada Petrovskiy bo'yicha parabolik tipdagi tenglama deyiladi.

Petrovskiy bo'yicha parabolik tipdagi tenglamaga eng sodda misol (uni parabolik tipdagi tenglama deb ham ataydilar) sifatida

$$\frac{\partial u}{\partial t} - \sum_{j=1}^{n-1} \frac{\partial^2 u}{\partial x_j^2} = 0$$

issiqlik o'tkazuvchanlik tenglamasini keltirish mumkin.

Haqiqatdan ham, $p=2$ zichlik bilan umumlashgan xarakteristik forma

$$\xi_n - \sum_{j=1}^{n-1} \xi_j^2$$

ko'rinishga ega bo'ladi. Shuningdek,

$$\sum_{|\alpha'|+p\alpha_n=m} a_\alpha(x)(i\xi')^{\alpha'} \xi_n^{\alpha_n} = 0$$

tenglama esa,

$$\xi_n - \sum_{j=1}^{n-1} \left(i\xi_j\right)^2 = 0$$

ko'rinishga ega bo'lib, $|\xi'|^2 = \sum_{j=1}^{n-1} \xi_j^2 = 1$ bo'ladigan ixtiyoriy haqiqiy $\xi' = (\xi_1, ..., \xi_{n-1})$ uchun ξ_n o'zgaruvchiga nisbatan yagona $\xi_n = -\sum_{j=1}^{n-1} \xi_j^2 = -1$ ildizga ega bo'ladi.

Xususiy hosilali yuqori tartibli sistemalarni klassifikatsiyalash ham matritsaviy differensial operatorning bosh qismini ajratish yo'li bilan amalga oshiriladi.

$n-$ o'lchamli R^n Evklid fazosidagi $\Omega \subset R^n$ sohada N noma'lumli N ta

$$L(x,D)u \equiv \sum_{j=1}^{N} \sum_{|\alpha|\le n_j} a_{ij}^{\alpha}(x) \frac{\partial^\alpha u_j}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}} = f_i(x), \quad i = 1,2,...,N \qquad (12)$$

chiziqli xususiy hosilali differensial tenglamalar sistemasi berilgan bo'lsin, bunda $u = (u_1, u_2, ..., u_N)$ noma'lum vektor–funksiya, $a_{ij}^{\alpha}(x)$ va $f_i(x)$ berilgan funksiyalar bo'lib, bu funksiyalar $\Omega \subset R^n$ sohada aniqlangan va uzluksizdir.

Ta'rif. *Agar $x^0 \in \Omega$ nuqtada (yoki $\Omega \subset R^n$ sohada) $|\xi| \ne 0$ bo'lgan har qanday $\xi = (\xi_1, \xi_2, ..., \xi_n)$ vektorlar uchun*

$$\det\left\| \sum_{|\alpha|=n_j} a_{ij}^{\alpha}(x^0)\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n} \right\|$$

determinant noldan farqli bo'lsa, u holda (12) sistema $x^0 \in \Omega$ *nuqtada (yoki* $\Omega \subset R^n$ *sohada) I.G. Petrovskiy ma'nosidagi elliptik sistema deyiladi*[2].

Bu ta'rifga ko'ra,

$$\begin{cases} \dfrac{\partial u}{\partial x_1} - \dfrac{\partial v}{\partial x_2} = 0, \\ \dfrac{\partial u}{\partial x_2} + \dfrac{\partial v}{\partial x_1} = 0 \end{cases}$$

Koshi–Riman sistemasi elliptik sistema bo'ladi, chunki $|\xi| \neq 0$ bo'lgan har qanday $\xi = (\xi_1, \xi_2)$ vektorlar uchun $\begin{vmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{vmatrix} = \xi_1^2 + \xi_2^2 \neq 0$ bo'ladi.

I.G. Petrovskiy ma'nosidagi elliptik sistemaning ta'rifiga ko'ra, umumiyroq bo'lgan, A. Duglis va L. Nirenbergga tegishli ta'rif ham mavjud[3].

Ushbu I.G. Petrovskiy ma'nosidagi elliptik sistemaning ta'rifida

$$L(x,D) \equiv \begin{pmatrix} l_{11}(x,D) & \dots & l_{1N}(x,D) \\ \vdots & \ddots & \vdots \\ l_{N1}(x,D) & \dots & l_{NN}(x,D) \end{pmatrix} \tag{13}$$

matritsaning har bir ustunidagi differensial operatorlar tartiblarining maksimal tartibi n_j sonni aniqlash kerak bo'ladi. Bu yerda $l_{ij}(x,D)$ differensial operatorlar $D = \left(\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_1} \right)$ differensial amalining o'zgaruvchi koeffitsientli polinomidan iborat, ya'ni har bir u_j, $j = 1,\dots,N$ noma'lum funksiya uchun

---

[2] Bu haqida Петровский И.Г., Об аналитичности решений систем уравнений с частными производными, Матем. сборник, 5(47): 1 (1939), С. 3–70., maqolasi orqali tanishish mumkin.
[3] Bu haqida Duglis A. and Nirenberg L., Interior estimates' for elliptic systems of partial differential equations, Comm. Pure Appl. Math., 8 (1955), P. 503–538., maqolasi orqali tanishish mumkin.

$$n_j = \max_{1\le i\le N} \deg l_{ij}(x,D),\ j=1,...,N$$

sonni aniqlaymiz, bunda deg simvoli orqali bu yerdagi operatorlar hosilalarining eng katta tartibi belgilangan. Bu belgilashga ko'ra, $l_{ij}(x,D)$ differensial operatorning n_j tartibli bosh qismini $\tilde{l}_{ij}(x,D)$ orqali belgilasak, u holda

$$\tilde{l}_{ij}(x,D) = \begin{cases} 0, & agar \quad \deg l_{ij}(x,D) < n_j \quad bo'lsa, \\ \sum\limits_{|\alpha|=n_j} a_{ij}^{\alpha}(x) D^{\alpha}, & agar \quad \deg l_{ij}(x,D) = n_j \quad bo'lsa \end{cases} \quad (14)$$

bo'ladi. Shunday qilib, $L(x,D)$ matritsaviy operatorning I.G. Petrovskiy ma'nosidagi elliptik sistema ekanligi $L(x,D)$ matritsaviy operatorning I.G. Petrovskiy bo'yicha bosh qismi bo'lgan

$$\tilde{L}(x,D) \equiv \begin{pmatrix} \tilde{l}_{11}(x,D) & \dots & \tilde{l}_{1N}(x,D) \\ \vdots & \ddots & \vdots \\ \tilde{l}_{N1}(x,D) & \dots & \tilde{l}_{NN}(x,D) \end{pmatrix} \quad (15)$$

matritsaviy operator bilan aniqlanadi, ya'ni agar $x^0 \in \Omega$ nuqtada (yoki $\Omega \subset R^n$ sohada) $|\xi| \ne 0$ bo'lgan har qanday $\xi = (\xi_1, \xi_2, ..., \xi_n)$ vektorlar uchun $\det \tilde{L}(x^0, \xi) \ne 0$ shart o'rinli bo'lsa, u holda (12) sistema $x^0 \in \Omega$ nuqtada (yoki $\Omega \subset R^n$ sohada) I.G. Petrovskiy ma'nosidagi elliptik sistema deyiladi[4].

A. Duglis va L. Nirenberg tomonidan kiritilgan Duglis–Nirenberg bo'yicha elliptik sistemani aniqlashda $L(x,D)$ matritsaviy operatorning bosh qismi boshqacha aniqlanadi va unda kichik tartibli hosilalar qatnashishi mumkin[5].

Butun sonlarning ikkita

---

[4] Batafsil I.G. Petrovskiy ma'nosidagi elliptik, giperbolik va parabolik sistemalar haqida ushbu И.Г. Петровский. Избранные труды. Системы уравнений с частными производными. Алгебраическая геометрия. М.: Наука, 1986г. стр. 500. nomli kitobi orqali tanishish mumkin.

[5] Bunday sistemalar va unga qo'yilgan umumiy chegaraviy shartli masalalarning yechilishi haqida В.А. Солонников, Об общих краевых задачах для систем, эллиптических в смысле А. Даглиса–Л. Ниренберга. I, Изв. АН СССР. Сер. матем., 1964, том 28, выпуск 3, С. 665–706., maqolasi orqali tanishish mumkin.

$$\{s_i\}_{i=1}^{N} \quad \text{va} \quad \{t_j\}_{j=1}^{N} \tag{16}$$

sistemalari quyidagi shartlarni qanoatlantirsin:

$$l_{ij}(x,D) \neq 0 \text{ bo‘lgan barcha } i,j \text{ uchun}$$

$$\deg l_{ij}(x,D) \leq s_i + t_j \tag{17}$$

va $s_i + t_j < 0$ bo‘lsa, u holda $l_{ij}(x,D) = 0$. Ko‘rinib turibdiki, bunday qo‘yilgan shartlarni qanoatlantiruvchi ixtiyoriy s_i, t_j sistemalar

$$s_i + const, \quad t_j - const$$

sistemalar bilan almashtirilishi mumkin. $L(x,D)$ matritsaviy operatorning Duglis–Nirenberg bo‘yicha bosh qismini (17) shartni qanoatlantiruvchi butun sonlarning oldindan berilgan (16) sistemalari yordamida aniqlanadi. Yuqorida kiritilgan sistemalarga ko‘ra, komponentalari butun sonlardan iborat bo‘lgan $s = (s_1, ..., s_N)$ va $t = (t_1, ..., t_N)$ vektorlarni qaraymiz. $L(x,D)$ matritsaviy operatorning s va t sistemalarga mos Duglis–Nirenberg bo‘yicha bosh qismi deb quyidagi matritsaviy differensial operatorga aytiladi:

$$\tilde{L}_{s,t}(x,D) \equiv \begin{pmatrix} \tilde{l}_{11}^{s,t}(x,D) & \cdots & \tilde{l}_{1N}^{s,t}(x,D) \\ \vdots & \ddots & \vdots \\ \tilde{l}_{N1}^{s,t}(x,D) & \cdots & \tilde{l}_{NN}^{s,t}(x,D) \end{pmatrix},$$

bunda $\tilde{l}_{ij}^{s,t}(x,D)$ differensial operatorlar

$$\tilde{l}_{ij}^{s,t}(x,D) = \begin{cases} 0, & agar \ \deg l_{ij}(x,D) < s_i + t_j \ bo'lsa, \\ & yoki \quad l_{ij}(x;D) = 0 \ bo'lsa, \\ \sum\limits_{|\alpha| = s_i + t_j} a_{ij}^{\alpha}(x) D^{\alpha}, & agar \ \deg l_{ij}(x,D) = s_i + t_j \ bo'lsa \end{cases}$$

formula yordamida aniqlanadi.

Ta’rif. *Agar (17) shartni qanoatlantiruvchi butun sonlarning oldindan berilgan ikkita sistemalari mavjud bo‘lib,*

buna $|\xi| \neq 0$ *bo‘lgan har qanday* $\xi = (\xi_1, \xi_2, ..., \xi_n)$ *vektorlar uchun*

$$\det \tilde{L}_{s,t}(x^0,\xi) \neq 0 \qquad (18)$$

shart o'rinli bo'lsa, u holda (12) sistema $x^0 \in \Omega$ *nuqtada Duglis–Nirenberg bo'yicha elliptik sistema deyiladi.*

Agar (12) sistema har bir $x^0 \in \Omega$ *nuqtada Duglis–Nirenberg bo'yicha elliptik sistema bo'lsa, u holda bu (12) sistema* Ω *sohada Duglis–Nirenberg bo'yicha elliptik sistema deyiladi.*

Bu yerda (18) shartga ko'ra,

$$L(x^0,\xi) = \det \tilde{L}_{s,t}(x^0,\xi) = \det\left\{\tilde{l}_{ij}^{s,t}(x^0,\xi)\right\}$$

polinom bo'lib, $|\xi| \neq 0$ bo'lgan har qanday $\xi = (\xi_1, \xi_2, ..., \xi_n)$ vektorlar uchun nolga teng emas, bunda $\tilde{l}_{ij}^{s,t}(x,D)$ differensial operator $l_{ij}^{s,t}(x,D)$ differensial operatorning $s_i + t_j$ tartibli aniqlikga ega bo'lgan bosh qismi. Shuningdek, $\xi = (\xi_1, \xi_2, ..., \xi_n)$ vektorga bog'liq bo'lgan $L(x^0,\xi)$ bir jinsli polinomning tartibi $\sum_{i=1}^{N}(s_i + t_j) = 2r$ songa teng, bunda r – manfiy bo'lmagan butun son. Biz $r > 0$ bo'lgan holni qaraymiz.

(17) shartlardan ko'rinadiki, bu yerda $s_i + t_j$ yig'indining qiymati muhim. Bu s_i va t_j sonlarni tanlashda qandaydir ma'noda ixtiyoriylikka ega bo'lamiz. Agar s vektorning barcha komponentalariga ayni bir sonni qo'shib va bu sonni t vektorning barcha komponentalaridan ayirsak, yangi s' va t' sistemalarni hosil qilamiz, ya'ni $s'_i = s_i - C$, $t'_j = t_j + C$. Bu s' va t' sistemalar

$$\tilde{L}_{s,t}(x^0,D) = \tilde{L}_{s',t'}(x^0,D)$$

tenglik ma'nosida s va t sistemalarga ekvivalent bo'ladi. Shuning uchun ham

$$s_i \leq 0,\ i = 1, 2, ..., N \qquad (19)$$

deb hisoblash mumkin. (12) sistema N nomalumli N ta tenglamalar sistemasi bo'lgani uchun har bir $j = 1, 2, ..., N$ uchun $i = 1, 2, ..., N$ ning hech bo'lmaganda bitta qiymati topiladiki, bunda $l_{ij}(x^0,D) \neq 0$

bo'ladi, ya'ni (13) matritsaning nolli ustunlari bo'lmaydi. U holda (17) shartdan

$$s_i + t_j \geq \deg l_{ij}(x, D) \geq 0 \qquad (20)$$

shart kelib chiqadi. (19) va (20) shartlardan

$$t_j \geq 0, \; j = 1, 2, \dots, N \qquad (21)$$

shartlarni hosil qilamiz. (19) va (21) shartlar s va t sistemalarning normalashtiruvchi shartlari deyiladi. (19) va (21) normalashtiruvchi shartlari muhim rol o'ynamasada, ular (17) va (18) shartlarni qanoatlantiruvchi s va t sistemalarni topish jarayonini yengillashtiradi. Shuningdek, $\max\limits_i s_i = 0$ deb olish qulay bo'ladi.

5-misol. Duglis–Nirenberg bo'yicha elliptik bo'lgan sistema va I.G. Petrovskiy bo'yicha elliptik bo'lmagan sistemaga muhim misol sifatida uch o'lchamli holda yopishqoq siqilmaydigan statsionar oqimni tavsiflaydigan chiziqlilashtirilgan Nave–Stoks tenglamalari

$$\begin{cases} -\Delta \mathbf{v} + grad\, p = 0, \\ div\, \mathbf{v} = 0 \end{cases} \qquad (22)$$

sistemani qaraymiz, bunda $\mathbf{v}(x) = (v_1(x), v_2(x), v_3(x))$ – tezliklar maydoni, $p(x)$ – skalyar funksiya.

Bu yerda $x \in \Omega \subset R^3$, $u = (v_1, v_2, v_3, p)$ belgilash kiritib, (22) sistemani $L(D)u(x) = 0$ ko'rinishida yozib olamiz, bunda

$$L(D) = \begin{pmatrix} -\Delta & 0 & 0 & \frac{\partial}{\partial x_1} \\ 0 & -\Delta & 0 & \frac{\partial}{\partial x_2} \\ 0 & 0 & -\Delta & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 \end{pmatrix} \qquad (23)$$

bo'ladi. Bu $L(D)$ operatorning Petrovskiy bo'yicha bosh qismini ajratamiz. Bu yerda $n_1 = n_2 = n_3 = 2,\ n_4 = 1$ bo'ladi. Shuning uchun $L(D)$ operatorning Petrovskiy bo'yicha bosh qismi

$$\tilde{L}(D) = \begin{pmatrix} -\Delta & 0 & 0 & \dfrac{\partial}{\partial x_1} \\ 0 & -\Delta & 0 & \dfrac{\partial}{\partial x_2} \\ 0 & 0 & -\Delta & \dfrac{\partial}{\partial x_3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

ko'rinishida bo'ladi. Bundan, har qanday $\xi = (\xi_1, \xi_2, \xi_3) \in R^3$ vektorlar uchun

$$\det \tilde{L}(\xi) = \begin{vmatrix} -|\xi|^2 & 0 & 0 & \xi_1 \\ 0 & -|\xi|^2 & 0 & \xi_2 \\ 0 & 0 & -|\xi|^2 & \xi_3 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

ekanligi kelib chiqadi, ya'ni (22) sistema Petrovskiy bo'yicha elliptik bo'lmaydi.

Bu (22) sistemaning Duglis–Nirenberg bo'yicha elliptik bo'lgan sistema ekanligini ko'rsatamiz. Buning uchun (17) va (18) shartlarni qanoatlantiruvchi s va t vektorlar sistemalari mavjudligini o'rnatish zarur. (19) va (21) normalashtiruvchi shartlaridan foydalanilsa, bu masala chekli sondagi variantlarni qarab chiqish bilan yechiladi: $L(D)$ operatorning bosh qismi

$$\tilde{L}_{s,t}(x,D) \equiv \begin{pmatrix} \tilde{l}_{11}^{s,t}(x,D) & \dots & \tilde{l}_{1N}^{s,t}(x,D) \\ \vdots & \ddots & \vdots \\ \tilde{l}_{N1}^{s,t}(x,D) & \dots & \tilde{l}_{NN}^{s,t}(x,D) \end{pmatrix}$$

matritsaviy differensial operatorga mos kelgan s_i va t_j sonlarning mumkin bo'lgan barcha qiymatlaridan s_i sonlar qiymatlari maksimal va

t_j sonlar qiymatlari minimal olinadi. Bunda, $\tilde{l}_{ij}(D)$ operator $l_{ij}(D)$ operatorning odatdagi bosh qismi, s_i sonlarning qiymatlarini $L(D)$ matritsadan chapda ustun ko‘rinishida, t_j sonlarning qiymatlarini $L(D)$ matritsaning yuqorisida satr ko‘rinishida yozib chiqish qulay. Qaralayotgan bu misolimizda (17) va (18) shartlarni qanoatlantiruvchi maksimal s_i va minimal t_j qiymatlar quyidagicha bo‘ladi:

$$s=(0,0,0,-1) \;\; \text{va} \;\; t=(2,2,2,1).$$

Shunga ko‘ra,

$$L(D)=\begin{pmatrix} -\Delta & 0 & 0 & \dfrac{\partial}{\partial x_1} \\ 0 & -\Delta & 0 & \dfrac{\partial}{\partial x_2} \\ 0 & 0 & -\Delta & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & 0 \end{pmatrix}$$

matritsaviy differensial operatorning mos bosh qismi

$$\tilde{L}_{s,t}(D)=\begin{pmatrix} -\Delta & 0 & 0 & \dfrac{\partial}{\partial x_1} \\ 0 & -\Delta & 0 & \dfrac{\partial}{\partial x_2} \\ 0 & 0 & -\Delta & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & 0 \end{pmatrix}$$

ko‘rinishda bo‘ladi. Uning xarakteristik determinanti esa, $|\xi|\neq 0$ bo‘lgan har qanday $\xi=(\xi_1,\xi_2,\xi_3)\in R^3$ vektorlar uchun

$$\det \tilde{L}_{s,t}(\xi) = \begin{vmatrix} -|\xi|^2 & 0 & 0 & \xi_1 \\ 0 & -|\xi|^2 & 0 & \xi_2 \\ 0 & 0 & -|\xi|^2 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{vmatrix} = -|\xi|^6 \neq 0$$

ekanligi kelib chiqadi, ya'ni (22) sistema Duglis–Nirenberg bo'yicha elliptik bo'lgan sistema bo'ladi.

6-misol. Istalgan elliptik bo'lgan sistema yoki tenglamani birinchi tartibli sistemaga keltirilishi mumkin. Tenglamalar va sistemalarni maxsusmas almashtirish bajarganda Petrovskiy bo'yicha elliptiklikni saqlamasa ham, Duglis–Nirenberg bo'yicha elliptikni saqlaydi. Bunga ishonch hosil qilish uchun tekislikda

$$\Delta u(x) = 0, \ x \in R^2$$

Laplas tenglamasini qaraymiz. Bu tenglama Petrovskiy bo'yicha elliptik bo'ladi. Bu ikkinchi tartibli tenglamani birinchi tartibli tenglamalar sistemasiga almashtiramiz. Agar

$$u_3(x) = u(x), \ u_1(x) = -\frac{\partial u_3}{\partial x_1}, \ u_2(x) = -\frac{\partial u_3}{\partial x_2}$$

belgilashlar kiritsak, u holda bu Laplas tenglamasini

$$\begin{cases} u_1 + \dfrac{\partial u_3}{\partial x_1} = 0, \\ u_2 + \dfrac{\partial u_3}{\partial x_2} = 0 \\ \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} = 0 \end{cases} \qquad (24)$$

ko'rinishidagi birinchi tartibli sistemaga almashtiramiz. Bu (24) sistema o'zgarmas koeffitsientli sistema bo'lib uning uchun (13) operator

$$L(D)=\begin{pmatrix} 1 & 0 & \frac{\partial}{\partial x_1} \\ 0 & 1 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & 0 \end{pmatrix} \qquad (25)$$

ko'rinishda bo'ladi. Bundan, (17) va (18) shartlarni qanoatlantiruvchi butun komponentali s va t vektorlarni topishga harakat qilamiz. Agar, misol uchun

$$s=(0,0,0),\ t=(1,1,1) \quad \text{yoki} \quad s=(0,0,0),\ t=(2,2,2).$$

sistemalar olinsa, ular (17) shartni qanoatlantiradi, lekin (18) shartni qanoatlantirmaydi, chunki bu sistemalar uchun (25) matritsaviy differensial operatorning bosh qismlari mos ravishda

$$\tilde{L}_{s,t}(D)=\begin{pmatrix} 0 & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & 0 \end{pmatrix} \quad \text{va} \quad \tilde{L}_{s,t}(D)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

ko'rinishda bo'lib, ular uchun (18) shart bajarilmaydi. Lekin, (17) shartni qanoatlantiruvchi $s=(-1,-1,0)$, $t=(1,1,2)$ sistemalar uchun mos bosh qism

$$\tilde{L}_{s,t}(D)=\begin{pmatrix} 1 & 0 & \frac{\partial}{\partial x_1} \\ 0 & 1 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & 0 \end{pmatrix}$$

ko'rinishda bo'ladi. Unga mos determinant esa, har qanday $\xi=(\xi_1,\xi_2)\in R^2\setminus\{0\}$ vektorlar uchun

$$\det \tilde{L}_{s,t}(\xi) = \begin{vmatrix} 1 & 0 & \xi_1 \\ 0 & 1 & \xi_2 \\ \xi_1 & \xi_2 & 0 \end{vmatrix} = -\xi_1^2 - \xi_2^2 \neq 0$$

ekanligi kelib chiqadi, ya'ni (18) shartni qanoatlantiradi. Shunga ko'ra, (24) sistema Duglis–Nirenberg bo'yicha elliptik bo'lgan sistema bo'ladi.

Endi (24) sistema uchun Petrovskiy bo'yicha bosh qismini aniqlaymiz, ya'ni (14) qoida bo'yicha (15) matritsaviy differensial operatorni aniqlaymiz. Bu bosh qism

$$\tilde{L}(D) = \begin{pmatrix} 0 & 0 & \dfrac{\partial}{\partial x_1} \\ 0 & 0 & \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & 0 \end{pmatrix}$$

ko'rinishda bo'ladi. Unga mos determinant esa, har qanday $\xi = (\xi_1, \xi_2) \in R^2 \setminus \{0\}$ vektorlar uchun

$$\det \tilde{L}(\xi) = \begin{vmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \\ \xi_1 & \xi_2 & 0 \end{vmatrix} = 0$$

ekanligi kelib chiqadi, ya'ni (24) sistema Petrovskiy bo'yicha elliptik bo'lmaydi.

Shunday qilib, tenglamalar va sistemalarni maxsusmas almashtirish Petrovskiy bo'yicha elliptikni saqlamas ekan.

Duglis–Nirenberg bo'yicha elliptik tenglamalarni va sistemalarni istalgan maxsusmas almashtirish natijasida ham saqlanishi M.F. Atya va I.M. Zingerlar tomonidan isbotlangan[6].

Shuni ta'kidlash kerakki, haqiqiy koeffitsientli Duglis–Nirenberg bo'yicha elliptik tenglamalar sistemasi uchun ξ bo'yicha (18) polinomning tartibi Petrovskiy sistemalari holidagi kabi, hamma vaqt juft

---

[6] Bu haqida Масленникова В.Н. Дифференциальные уравнения в частных производных. Москва: Издательство Российского университета дружбы народов, 1997. 447 с., kitobi orqali tanishish mumkin.

bo'ladi. Lekin, agar kompleks koeffitsientli sistemaga ega bo'lsak, u holda polinomning tartibi erkli o'zgaruvchilar soni $n \geq 3$ bo'lganda juft bo'ladi; $n = 2$ bo'lgan holda (18) polinomning tartibi juft bo'lishi ham va toq bo'lishi ham mumkin[7].

7-misol. Koshi–Riman sistemasining uch o'lchamli fazodagi analogi deb hisoblash mumkin bo'lgan Mosil–Teodoresko (MT)[8]

$$\begin{pmatrix} 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix} (s,u,v,w) = 0$$

sistemani elliptiklikka tekshiramiz. Ushbu Mosil–Teodoresko sistemasining differensial operatori

$$L(D) = \begin{pmatrix} 0 & \partial_x & \partial_y & \partial_z \\ \partial_x & 0 & -\partial_z & \partial_y \\ \partial_y & \partial_z & 0 & -\partial_x \\ \partial_z & -\partial_y & \partial_x & 0 \end{pmatrix}$$

ko'rinishda bo'lib, uning Petrovskiy bo'yicha bosh qismi o'ziga teng, ya'ni $\tilde{L}(D) = L(D)$ bo'ladi. Bu sistemaning xarakteristik determinanti har qanday $\xi = (\xi_1, \xi_2, \xi_3) \in R^3 \setminus \{0\}$ vektorlar uchun

---

[7] Bu haqida Agmon S. Lectures on elliptic boundary value problems. Princeton, 1965, kitobi orqali tanishish mumkin.

[8] Bunday elliptik sistemalarni kengroq o'rganish uchun 1) Янушаускас А.И. Задача о наклонной производной теория потенциала. Новосибирск: Наука СО, 1985. 262 с., 2) Янушаускас А.И. Методы потенциала в теории эллиптических уравнений. Вильнюс: Мокслас, 1990. 264 с., hamda 3) Ishankulov T. Elliptik tenglamalar va taqsimotlar nazariyasi. Samarqand: SamDU, 2019, 194 bet, kabi o'quv qo'llanmalari orqali ham tanishish mumkin.

$$\det L(\xi)=\begin{vmatrix}0 & \xi_1 & \xi_2 & \xi_3\\ \xi_1 & 0 & -\xi_3 & \xi_2\\ \xi_2 & \xi_3 & 0 & -\xi_1\\ \xi_3 & -\xi_2 & \xi_1 & 0\end{vmatrix}=-(\xi_1^2+\xi_2^2+\xi_3^2)^2=-|\xi|^4\neq 0$$

ekanligi kelib chiqadi. Shunga ko'ra, Mosil–Teodoresko (MT) sistemasi Petrovskiy bo'yicha elliptik bo'ladi.

Shuningdek, Koshi–Riman sistemasining to'rt o'lchamli fazodagi analogi deb hisoblash mumkin bo'lgan sistema ushbu

$$\begin{pmatrix}\frac{\partial}{\partial t} & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial t} & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial t} & -\frac{\partial}{\partial x}\\ \frac{\partial}{\partial z} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & \frac{\partial}{\partial t}\end{pmatrix}(s,u,v,w)=0$$

ko'rinishida bo'ladi.

8-misol. Elastiklik nazariyasining tenglamalar sistemasi vektor formada

$$\mu\Delta\mathbf{u}(x)+(\lambda+\mu)\,grad\,div\,\mathbf{u}(x)=0,\quad x\in D\subset R^3 \tag{26}$$

ko'rinishida bo'ladi, bunda $\lambda>0,\ \mu>0$. Bu tenglamaning koordinatalardagi ifodasi

$$\begin{cases}\mu\Delta u_1+(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_2}{\partial x_1\partial x_2}+\frac{\partial^2 u_3}{\partial x_1\partial x_3}\right)=0,\\ \mu\Delta u_2+(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x_1\partial x_2}+\frac{\partial^2 u_2}{\partial x_2^2}+\frac{\partial^2 u_3}{\partial x_2\partial x_3}\right)=0,\\ \mu\Delta u_3+(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x_3\partial x_1}+\frac{\partial^2 u_2}{\partial x_3\partial x_2}+\frac{\partial^2 u_3}{\partial x_3^2}\right)=0\end{cases} \tag{27}$$

ko'rinishda bo'ladi. Ushbu (27) sistemaning matritsaviy differensial operatori

$$L(D)=\begin{pmatrix} \mu\Delta+(\lambda+\mu)\dfrac{\partial^2}{\partial x_1^2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_1\partial x_2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_1\partial x_3} \\ (\lambda+\mu)\dfrac{\partial^2}{\partial x_1\partial x_2} & \mu\Delta+(\lambda+\mu)\dfrac{\partial^2}{\partial x_2^2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_2\partial x_3} \\ (\lambda+\mu)\dfrac{\partial^2}{\partial x_3\partial x_1} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_3\partial x_2} & \mu\Delta+(\lambda+\mu)\dfrac{\partial^2}{\partial x_3^2} \end{pmatrix}$$

ko'rinishda bo'lib, uning xarakteristik formasi quyidagicha bo'ladi:

har qanday $\xi=(\xi_1,\xi_2,\xi_3)\in R^3\setminus\{0\}$ vektorlar uchun

$$\det \tilde{L}(\xi)=\det L(\xi)=$$

$$=\begin{vmatrix} \mu|\xi|^2+(\lambda+\mu)\xi_1^2 & (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_1\xi_3 \\ (\lambda+\mu)\xi_1\xi_2 & \mu|\xi|^2+(\lambda+\mu)\xi_2^2 & (\lambda+\mu)\xi_2\xi_3 \\ (\lambda+\mu)\xi_1\xi_3 & (\lambda+\mu)\xi_2\xi_3 & \mu|\xi|^2+(\lambda+\mu)\xi_3^2 \end{vmatrix}=$$

$$=\left[\mu|\xi|^2+(\lambda+\mu)\xi_1^2\right]\left[\mu|\xi|^2+(\lambda+\mu)\xi_2^2\right]\left[\mu|\xi|^2+(\lambda+\mu)\xi_3^2\right]+$$

$$+(\lambda+\mu)^3\xi_1^2\xi_2^2\xi_3^2+(\lambda+\mu)^3\xi_1^2\xi_2^2\xi_3^2-\mu(\lambda+\mu)^2|\xi|^2\xi_1^2\xi_3^2-$$

$$-(\lambda+\mu)^3\xi_1^2\xi_2^2\xi_3^2-\mu(\lambda+\mu)^2|\xi|^2\xi_2^2\xi_3^2-(\lambda+\mu)^3\xi_1^2\xi_2^2\xi_3^2-$$

$$-\mu(\lambda+\mu)^2|\xi|^2\xi_1^2\xi_2^2-(\lambda+\mu)^3\xi_1^2\xi_2^2\xi_3^2=$$

$$=\left[\mu|\xi|^2+(\lambda+\mu)\xi_1^2\right]\left[\mu|\xi|^2+(\lambda+\mu)\xi_2^2\right]\left[\mu|\xi|^2+(\lambda+\mu)\xi_3^2\right]-$$

$$-\mu(\lambda+\mu)^2|\xi|^2\left[\xi_1^2\xi_3^2+\xi_2^2\xi_3^2+\xi_1^2\xi_2^2\right]-(\lambda+\mu)^3\xi_1^2\xi_2^2\xi_3^2=$$

$$=\mu^3|\xi|^6+\mu^2(\lambda+\mu)|\xi|^4\left[\xi_1^2+\xi_2^2+\xi_3^2\right]+$$

$$+\mu(\lambda+\mu)^2|\xi|^2\left[\xi_1^2\xi_2^2+\xi_1^2\xi_3^2+\xi_2^2\xi_3^2\right]+(\lambda+\mu)^3\xi_1^2\xi_2^2\xi_3^2-$$

$$-\mu(\lambda+\mu)^2|\xi|^2\left[\xi_1^2\xi_3^2+\xi_2^2\xi_3^2+\xi_1^2\xi_2^2\right]-(\lambda+\mu)^3\xi_1^2\xi_2^2\xi_3^2=$$

$$=\mu^3|\xi|^6+\mu^2(\lambda+\mu)|\xi|^6=\mu^2(\lambda+2\mu)|\xi|^6>0$$

ekanligi kelib chiqadi. Shunga ko'ra, elastiklik nazariyasining tenglamalar sistemasi Petrovskiy bo'yicha elliptik sistema bo'ladi.

Ta'rif. Agar $w = w(z)$ funksiya kompleks tekislikdagi biror D sohada $\frac{\partial^2 w(z)}{\partial \bar{z}^2} = 0$ tenglamani qanoatlantirsa, u holda $w(z)$ funksiya D sohada bianalitik funksiya deyiladi.

9-misol. Tekislikda bianalitik funksiyaning haqiqiy va mavhum qismlari

$$u_{xx} - u_{yy} - 2v_{xy} = 0, \quad v_{xx} - v_{yy} + 2u_{xy} = 0 \tag{28}$$

ikkinchi tartibli elliptik sistemani qanoatlantirishini ko'rsatamiz.

Haqiqatdan ham, agar

$$w(z) = f(z) = u(x,y) + iv(x,y)$$

funksiya $\mathbb{C}$ kompleks tekislikning qismi bo'lgan biror D sohada bianalitik funksiya bo'lsa, u holda

$$f_{\bar{z}}(z) = \frac{1}{2}\left((u+iv)_x + i(u+iv)_y\right) = \frac{1}{2}\left(u_x - v_y + i(u_y + v_x)\right),$$

$$f_{\bar{z}\bar{z}}(z) = \frac{1}{4}\left[\left(u_x - v_y + i(u_y + v_x)\right)_x + i\left(u_x - v_y + i(u_y + v_x)\right)_y\right] =$$

$$= \frac{1}{4}\left[\left(u_{xx} - u_{yy} - 2v_{xy}\right) + i\left(v_{xx} - v_{yy} + 2u_{xy}\right)\right]$$

bo'lgani uchun

$$\frac{1}{4}\left[\left(u_{xx} - u_{yy} - 2v_{xy}\right) + i\left(v_{xx} - v_{yy} + 2u_{xy}\right)\right] = 0 \tag{29}$$

kompleks tenglikdan (28) sistemani hosil qilamiz.

10-misol. Bianalitik funksiyaning haqiqiy va mavhum qismlari bigarmonik funksiyalar bo'ladi, ya'ni $u(x,y)$ va $v(x,y)$ funksiyalar uchun

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0, \tag{30}$$

$$\frac{\partial^4 v}{\partial x^4} + 2\frac{\partial^4 v}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial y^4} = 0 \tag{30'}$$

tengliklar o'rinli bo'ladi. Haqiqatdan ham, agar (28) tenglamalar sistemasining birinchi tenglamasiga $\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ operatorni, ikkinchi tenglamasiga $2\frac{\partial^2}{\partial x \partial y}$ operatorni ta'sir qildirib, hosil bo'lgan tenglamalarni hadma–had qo'shsak, u holda $f(z)$ bianalitik funksiyaning haqiqiy qismi bo'lgan $u = u(x,y)$ funksiya (30) tenglamani qanoatlantirishi kelib chiqadi. Xuddi shuningdek, agar (28) tenglamalar sistemasining birinchi tenglamasiga $-2\frac{\partial^2}{\partial x \partial y}$ operatorni, ikkinchi tenglamasiga $\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ operatorni ta'sir qildirib, hosil bo'lgan tenglamalarni hadma–had qo'shsak, u holda $f(z)$ bianalitik funksiyaning mavhum qismi bo'lgan $v = v(x,y)$ funksiya (30') tenglamani qanoatlantirishi kelib chiqadi. Bu (28) tenglamalar sistemasining Petrovskiy bo'yicha elliptik sistema ekanligini ham ko'rish mumkin.

11-misol. Tekislikdagi biror $G \subset \mathbb{C}$ bir bog'lamli sohada bigarmonik bo'lgan, ya'ni $n = 2$, $z = x + iy \in G$ bo'lgan hol uchun $\Delta^2 u(x,y) = 0$ tenglamaning shu sohadagi yechimlari $u(x,y) = \mathrm{Re}\left[\bar{z}\varphi(z) + \psi(z)\right]$ shaklda tasvirlanishini ko'rsatamiz. Bu yerda $\varphi(z)$ va $\psi(z)$ funksiyalar ixtiyoriy analitik funksiyalardir. Haqiqatdan ham, bigarmonik bo'lgan $\Delta^2 u(x,y) = 0$ tenglamani $\frac{\partial^4 u}{\partial z^2 \partial \bar{z}^2} = 0$ ko'rinishida yozib, uni integrallasak,

$$\frac{\partial^3 u}{\partial z^2 \partial \bar{z}} = \varphi_1(z), \quad \frac{\partial^2 u}{\partial z^2} = \bar{z}\varphi_1(z) + \psi_1(z)$$

bo'ladi, bunda $\varphi_1(z)$ va $\psi_1(z)$ funksiyalar G sohadagi z o'zgaruvchiga nisbatan ixtiyoriy analitik funksiyalardir. Integrallashni davom ettirib,

$$\frac{\partial u}{\partial z} = \bar{z}\varphi_2(z) + \psi_2(z) + \overline{h}_1(\bar{z})$$

tenglikni yozamiz, bunda $\varphi_2(z) = \int\limits_{z_0}^{z} \varphi_1(\zeta)d\zeta$ va $\psi_2(z) = \int\limits_{z_0}^{z} \psi_1(\zeta)d\zeta$ – boshlang'ich funksiyalar bo'lib, G sohadagi analitik funksiyalardir, $\overline{h}_1$ –

esa, $\bar{z}=x-iy$ o'zgaruvchiga nisbatan ixtiyoriy analitik funksiyadir. Shunga ko'ra, h_1 funksiya z o'zgaruvchiga nisbatan analitik funksiya bo'ladi. Shuningdek,

$$u=\bar{z}\varphi_3(z)+\psi_3(z)+z\bar{h}_1+\bar{h}_2$$

tenglik o'rinli, bunda $\varphi_3(z)=\int\limits_{z_0}^{z}\varphi_2(\zeta)d\zeta$ va $\psi_3(z)=\int\limits_{z_0}^{z}\psi_2(\zeta)d\zeta$ – boshlang'ich funksiyalar bo'lib, G sohadagi analitik funksiyalardir, $\bar{h}_2$ – esa, $\bar{z}=x-iy$ o'zgaruvchiga nisbatan ixtiyoriy analitik funksiyadir. Shunga ko'ra, h_2 funksiya z o'zgaruvchiga nisbatan analitik funksiya bo'ladi. Demak, $u(x,y)$ – haqiqiy qiymatli bigarmonik funksiya bo'lgani uchun

$$u(x,y)=\mathrm{Re}\left(\bar{z}\varphi_3(z)+\psi_3(z)\right)+\mathrm{Re}\left(z\bar{h}_1+\bar{h}_2\right)=$$
$$=\mathrm{Re}\left(\bar{z}\varphi_3(z)+\psi_3(z)\right)+\mathrm{Re}\left(\overline{\bar{z}h_1+h_2}\right)=$$
$$=\mathrm{Re}\left[\bar{z}\left(\varphi_3(z)+h_1(z)\right)+\left(\psi_3(z)+h_2(z)\right)\right]$$

tenglik hosil bo'ladi. Agar $\varphi(z)=\varphi_3(z)+h_1(z)$ va $\psi(z)=\psi_3(z)+h_2(z)$ deb belgilasak, u holda

$$u(x,y)=\mathrm{Re}\left[\bar{z}\varphi(z)+\psi(z)\right] \tag{31}$$

tenglikni yozamiz. Bu yerda $\varphi(z)$ va $\psi(z)$ funksiyalar ixtiyoriy analitik funksiyalardir.

Shuningdek, har qanday $u(x_1,...,x_n)$ bigarmonik funksiyani

$$u(x_1,...,x_n)=\left(x_1^2+x_2^2+...+x_n^2\right)\cdot v_1(x_1,...,x_n)+v_2(x_1,...,x_n)$$

shaklda ham yozish mumkin, bu yerda $v_1(x_1,...,x_n)$ va $v_2(x_1,...,x_n)$ garmonik funksiyalar.

11. Aralash tipdagi ikki o'zgaruvchili ikkinchi tartibli kvazichiziqli differensial tenglamalarni kanonik ko'rinishga keltirish. Ushbu

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0 \tag{32}$$

kvazichiziqli tenglamani qaraylik. Agar (32) tenglamani nuqta atrofida emas, balki butun bir D sohada qarasak, u holda yuqorida ko'rsatilgan ikkinchi tartibli tenglamalar klassifikatsiyasining uch tipi to'laligicha javob bermaydi, chunki B^2-AC ifoda umuman olganda, butun sohada ishorasini o'zgartirishi mumkin. Shuning uchun, tenglamalarning xarakteristikalari qisman haqiqiy va qisman mavhum bo'lishi mumkin.

Demak, agar B^2-AC ifoda ishorasini D sohada o'zgartirsa, u holda (32) tenglama *aralash tipdagi tenglama* deb ataladi. D sohada $B^2-AC=0$ tenglama bilan aniqlanadigan γ chiziq (32) tenglamaning *parabolik chizig'i* deb ataladi, mos ravishda B^2-AC ifodaning ishorasi D sohada $B^2-AC<0$ bo'lganda elliptik qismi va D sohada $B^2-AC>0$ bo'lganda giperbolik qismi deb ataladi.

Ushbu D sohada

$$\xi=\xi(x,y),\quad \eta=\eta(x,y) \tag{33}$$

yangi o'zgaruvchilarni kiritamiz. U holda yangi o'zgaruvchilarga nisbatan (32) tenglama

$$\bar{A}(\xi,\eta)\frac{\partial^2 u}{\partial\xi^2}+2\bar{B}(\xi,\eta)\frac{\partial^2 u}{\partial\xi\partial\eta}+\bar{C}(\xi,\eta)\frac{\partial^2 u}{\partial\eta^2}+F\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)=0 \tag{34}$$

shaklida yoziladi, bunda

$$\bar{A}(\xi,\eta)=A\left(\frac{\partial\xi}{\partial x}\right)^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C\left(\frac{\partial\xi}{\partial y}\right)^2,$$

$$\bar{B}(\xi,\eta)=A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}, \tag{35}$$

$$\bar{C}(\xi,\eta)=A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2.$$

Tanlash bizning ixtiyorimizda bo'lgan ikkita $\xi=\xi(x,y)$, $\eta=\eta(x,y)$ funksiyalarni

$$A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}=0, \tag{36}$$

$$A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2\neq 0 \qquad (37)$$

shartlar bajariladigan qilib tanlaymiz. Shuningdek, γ parabolik chiziq ustida $AC-B^2=0$ ifodani

$$AC-B^2=H^n(x,y)M(x,y) \qquad (38)$$

shaklida tasvirlash mumkin, bunda D sohada $M(x,y)\neq 0$, hamda $H(x,y)=0$ tenglama γ chiziqning tenglamasi bo'lib, $\frac{\partial H(x,y)}{\partial x}$ va $\frac{\partial H(x,y)}{\partial y}$ hosilalar bir vaqtda nolga teng emas.

Quyidagi ikki holni qaraymiz:

1–hol. γ parabolik chiziq nuqtalarida (32) tenglama xarakteristikasining yo'nalishi shu chiziq urinmasining yo'nalishi bilan ustma–ust tushmaydi, ya'ni γ chiziq bo'ylab,

$$A\left(\frac{\partial H}{\partial x}\right)^2+2B\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}+C\left(\frac{\partial H}{\partial y}\right)^2\neq 0 \qquad (39)$$

shart bajariladi. Bu holda

$$\eta=H(x,y) \qquad (40)$$

deb olamiz. Shuningdek, $\xi=\xi(x,y)$ funksiya sifatida

$$\left(A\frac{\partial H}{\partial x}+B\frac{\partial H}{\partial y}\right)\frac{\partial\xi}{\partial x}+\left(B\frac{\partial H}{\partial x}+C\frac{\partial H}{\partial y}\right)\frac{\partial\xi}{\partial y}=0 \qquad (41)$$

tenglamaning yechimini olamiz. Ushbu $\xi=\xi(x,y)$, $\eta=\eta(x,y)$ funksiyalarni bunday tanlashda, (36) va (37) shartlar bajariladi. Bu funksiyalarning yakobiani γ chiziq atrofida nolga teng emasligini ko'rsatamiz. Haqiqatdan ham, agar

$$\frac{\partial\xi}{\partial x}=\rho\left(B\frac{\partial H}{\partial x}+C\frac{\partial H}{\partial y}\right),\quad \frac{\partial\xi}{\partial y}=-\rho\left(A\frac{\partial H}{\partial x}+B\frac{\partial H}{\partial y}\right), \qquad (42)$$

bunda γ chiziq bo'ylab $\rho(x,y)\neq 0$ deb olsak, u holda shu γ chiziq bo'ylab,

$$\frac{D(\xi,\eta)}{D(x,y)}=\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}=$$

$$=\rho\left[\left(B\frac{\partial H}{\partial x}+C\frac{\partial H}{\partial y}\right)\frac{\partial H}{\partial y}+\left(A\frac{\partial H}{\partial x}+B\frac{\partial H}{\partial y}\right)\frac{\partial H}{\partial x}\right]=$$

$$=\rho\left[A\left(\frac{\partial H}{\partial x}\right)^2+2B\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}+C\left(\frac{\partial H}{\partial y}\right)^2\right]\neq 0$$

munosabat o'rinli bo'ladi. A,B,C va H funksiyalarning uzluksizligiga ko'ra, yakobian γ chiziqning qandaydir atrofida ham nolga teng bo'lmaydi. Shuning uchun, bu atrofda (33) tenglikdagi almashtirishni

$$\xi=\xi(x,y),\quad \eta=\eta(x,y)=H(x,y)$$

deb qabul qilamiz.

U holda (34) tenglamaning chap tomonidagi $\overline{B}(\xi,\eta)=0$ ekanligi (35) va (41) munosabatlardan kelib chiqadi. Shuningdek, γ chiziqning qandaydir atrofida $\overline{C}(\xi,\eta)\neq 0$ bo'ladi. Hosil qilingan (34) tenglamaning har ikkala tomonini $\overline{C}(\xi,\eta)\neq 0$ koeffitsientga bo'lib,

$$\frac{\overline{A}(\xi,\eta)}{\overline{C}(\xi,\eta)}\cdot\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=F_1\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)$$

tenglamani hosil qilamiz, yoki (35), (38), (40) va (42) munosabatlarni e'tiborga olib,

$$\eta^n K_1(\xi,\eta)\cdot\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=F_1\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)\quad(43)$$

tenglamani hosil qilamiz, bunda γ chiziqning qandaydir atrofida $K_1(\xi,\eta)\neq 0$ bo'ladi.

2–hol. γ parabolik chiziq xarakteristika bo'ladi yoki (32) tenglamaning xarakteristikalari oilasining o'ramasi bo'ladi, ya'ni γ chiziqning barcha nuqtalarida

$$A\left(\frac{\partial H}{\partial x}\right)^2+2B\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}+C\left(\frac{\partial H}{\partial y}\right)^2=0\quad(44)$$

tenglik o'rinli bo'ladi.

Aytaylik, γ chiziq bo'ylab, $A \geq 0, C \geq 0$ bo'lsin. U holda $B^2 - AC = 0$ shartga ko'ra, (44) tenglikni

$$\sqrt{A}\frac{\partial H}{\partial x} + \varepsilon\sqrt{C}\frac{\partial H}{\partial y} = 0 \qquad (45)$$

shaklida yozish mumkin, bunda $\varepsilon = signB$ songa teng bo'ladi. Agar $B = 0$ bo'lsa, u holda γ chiziq bo'ylab yoki $A = 0$, yoki $C = 0$ bo'ladi va (45) tenglikdan $H_y = 0$ yoki $H_x = 0$ tengliklarga ega bo'lamiz.

$\eta = \eta(x, y)$ funksiya sifatida

$$n(x,y)\frac{\partial \eta}{\partial x} - m(x,y)\frac{\partial \eta}{\partial y} = 0 \qquad (46)$$

tenglamaning yechimini olamiz, bunda $n = n(x, y)$ va $m = m(x, y)$ funksiyalar γ chiziqning atrofida

$$Am^2 + 2Bmn + Cn^2 \neq 0 \qquad (47)$$

shartni qanoatlantiradi. Masalan, agar D sohada $A \neq 0$ yoki $C \neq 0$ bo'lsa, u holda $m = 1$, $n = 0$ uchun $\eta = x$ deb, yoki $m = 0$, $n = 1$ uchun $\eta = y$ deb olish mumkin bo'ladi.

$\xi = \xi(x, y)$ funksiya sifatida esa,

$$\left(A\frac{\partial \eta}{\partial x} + B\frac{\partial \eta}{\partial y}\right)\frac{\partial \xi}{\partial x} + \left(B\frac{\partial \eta}{\partial x} + C\frac{\partial \eta}{\partial y}\right)\frac{\partial \xi}{\partial y} = 0 \qquad (48)$$

tenglamaning yechimini olamiz, bunda $\eta = \eta(x, y)$ funksiya (46) tenglamaning yechimidir.

Shunga ko'ra, $\xi = \xi(x, y)$ va $\eta = \eta(x, y)$ funksiyalarning bunday tanlanishida (36) va (37) shartlar bajariladi.

(48) tenglamaning γ chiziq va $\eta(x, y) = 0$ chiziqlarning kesishish nuqtasida $\xi(x, y) = 0$ bo'ladigan yechimini hamma vaqt tanlab olish mumkin ekanligini ta'kidlash kerak.

Xuddi birinchi holdagidek, γ chiziq bo'ylab, yakobian $\dfrac{D(\xi,\eta)}{D(x,y)} \neq 0$ ekanligini ko'rsatish qiyin emas. U holda A, B, C, m va n funksiyalarning uzluksizligiga ko'ra, bu yakobian γ chiziqning qandaydir atrofida ham, noldan farqli bo'ladi.

Endi $H(x,y)=\overline{H}(\xi,\eta)$ deb olamiz. U holda

$$\frac{\partial\overline{H}}{\partial\xi}=\frac{\partial H}{\partial x}\frac{\partial x}{\partial\xi}+\frac{\partial H}{\partial y}\frac{\partial y}{\partial\xi}=\frac{H_x\eta_y-H_y\eta_x}{\xi_x\eta_y-\xi_y\eta_x},\tag{49}$$

$$\frac{\partial\overline{H}}{\partial\eta}=\frac{\partial H}{\partial x}\frac{\partial x}{\partial\eta}+\frac{\partial H}{\partial y}\frac{\partial y}{\partial\eta}=-\frac{H_x\xi_y-H_y\xi_x}{\xi_x\eta_y-\xi_y\eta_x}$$

tengliklarni hosil qilamiz. Shuningdek, γ chiziq bo'ylab $\frac{\partial\overline{H}}{\partial\xi}\neq 0$ bo'ladi, chunki $\eta=const$ chiziq hech bir joyda γ chiziq bilan urinmaydi. (45) tenglikdan γ chiziq bo'ylab

$$\frac{\partial H}{\partial x}=-\sigma\varepsilon\sqrt{C},\ \frac{\partial H}{\partial y}=\sigma\sqrt{A}\quad[\sigma(x,y)\neq 0],$$

$$\frac{\partial\overline{H}}{\partial\eta}=\frac{\sigma}{\xi_x\eta_y-\xi_y\eta_x}\left(\sqrt{A}\frac{\partial\xi}{\partial x}+\varepsilon\sqrt{C}\frac{\partial\xi}{\partial y}\right)=0\tag{50}$$

tengliklar kelib chiqadi, chunki (48) tenglama γ chiziq bo'ylab, $B^2-AC=0$ shartdan

$$\sqrt{A}\frac{\partial\xi}{\partial x}+\varepsilon\sqrt{C}\frac{\partial\xi}{\partial y}=0$$

ko'rinishga keladi.

$\overline{H}(\xi,\eta)=0$ tenglikdan $\frac{\partial\xi}{\partial\eta}=-\frac{\overline{H}_\eta}{\overline{H}_\xi}$ tenglikka ega bo'lamiz va (50) tengliklardan γ chiziq bo'ylab, $\xi=const$ ekanligi kelib chiqadi. Lekin, γ chiziq bilan kesishish nuqtasida $\xi(x,y)=0$ bo'ladi va shunga ko'ra, γ chiziq bo'ylab, $\xi=0$ ekanligi kelib chiqadi. Shuning uchun, $\overline{H}(0,\eta)=0$ va

$$\overline{H}(\xi,\eta)=\xi\overline{H}_\xi(\theta(\xi,\eta)\xi,\eta)=\xi N(\xi,\eta)\tag{51}$$

tenglikni yozishimiz mumkin bo'ladi, bunda $0<\theta(\xi,\eta)<1$ va $N(\xi,\eta)\neq 0$.

Yangi o'zgaruvchilarga almashtirilgan (34) tenglamada γ chiziqning qandaydir atrofida $\overline{B}=0$ va $\overline{C}\neq 0$ ekanligi (36) va (37) munosabatlardan kelib chiqadi. (34) tenglamaning har ikkala tomonini $\overline{C} \neq 0$ koeffitsientga bo'lib,

$$\frac{\overline{A}(\xi,\eta)}{\overline{C}(\xi,\eta)} \cdot \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_2\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)$$

tenglamani hosil qilamiz, lekin

$$\frac{\overline{A}(\xi,\eta)}{\overline{C}(\xi,\eta)} = \frac{A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2}{A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2} = \rho^2\left(AC - B^2\right) =$$

$$= \rho^2 H^n(x,y) M(x,y) = \xi^n \rho^2 N^n M = \xi^n K_2(\xi,\eta)$$

bo'lgani uchun

$$\xi^n K_2(\xi,\eta) \cdot \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_2\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) \qquad (52)$$

ko'rinishdagi tenglamani hosil qilamiz, bunda γ chiziqning qandaydir atrofida $K_2(\xi,\eta) \neq 0$ bo'ladi.

1–misol.

$$(1-x^2)\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} - (1+y^2)\frac{\partial^2 u}{\partial y^2} - 2x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$$

tenglamani qaraymiz. Bu tenglama aralash tipdagi tenglamadir, chunki $AC - B^2 = x^2 - y^2 - 1 = H(x,y)$ bo'ladi. Bu yerda $1 - x^2 + y^2 > 0$ sohada berilgan tenglama giperbolik tipdagi va $1 - x^2 + y^2 < 0$ sohada berilgan tenglama elliptik tipdagi tenglama bo'ladi. Hamda, $x^2 - y^2 = 1$ chiziq berilgan tenglamaning parabolik chizig'i bo'ladi. Shuningdek, γ chiziq bo'ylab,

$$A\left(\frac{\partial H}{\partial x}\right)^2 + 2B\frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + C\left(\frac{\partial H}{\partial y}\right)^2 =$$

$$= 4x^2(1-x^2) + 8x^2y^2 - 4y^2(1+y^2) = 4(y^2 - x^2)(x^2 - y^2 - 1) = 0$$

bo'ladi. Shunga ko'ra, ikkinchi holga ega bo'lamiz. Umumiy nazariyaga ko'ra, $\xi = \xi(x,y)$ va $\eta = \eta(x,y)$ funksiyalarni (46) va (48) tenglamalarning yechimlaridan tanlab olamiz. Masalan, $n=1+x$, $m=-y$ deb olamiz. U holda (46) tenglama

$$(1+x)\frac{\partial\eta}{\partial x}+y\frac{\partial\eta}{\partial y}=0$$

ko'rinishida bo'ladi. Bu tenglama uchun $\eta=\frac{y}{1+x}$ funksiya xususiy yechim bo'ladi. Endi $\eta=\frac{y}{1+x}$ funksiyani (48) tenglamaga qo'yamiz. U holda $y(1+x)\frac{\partial\xi}{\partial x}+(1+x+y^2)\frac{\partial\xi}{\partial y}=0$ tenglama hosil bo'ladi. Bu tenglama uchun $\xi(x,y)=\frac{x^2-y^2-1}{(1+x)^2}$ funksiya xususiy yechim bo'ladi.

Demak, yangi ξ va η o'zgaruvchilarni

$$\xi(x,y)=\frac{x^2-y^2-1}{(1+x)^2}, \quad \eta=\frac{y}{1+x}$$

formulalar bo'yicha kiritamiz. U holda tegishli hosilalarni hisoblab,

$$\frac{\partial u}{\partial x}=\frac{1+x+y^2}{2(1+x)^3}\frac{\partial u}{\partial\xi}-\frac{y}{(1+x)^2}\frac{\partial u}{\partial\eta},$$

$$\frac{\partial u}{\partial y}=-\frac{y}{2(1+x)^2}\frac{\partial u}{\partial\xi}+\frac{1}{1+x}\frac{\partial u}{\partial\eta},$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{(1+x+y^2)^2}{4(1+x)^6}\frac{\partial^2 u}{\partial\xi^2}-\frac{y(1+x+y^2)}{(1+x)^5}\frac{\partial^2 u}{\partial\xi\partial\eta}+$$

$$+\frac{y^2}{(1+x)^4}\frac{\partial^2 u}{\partial\eta^2}-\frac{1+x+3y^2}{2(1+x)^4}\frac{\partial u}{\partial\xi}+\frac{2y}{(1+x)^3}\frac{\partial u}{\partial\eta},$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{y^2}{4(1+x)^4}\frac{\partial^2 u}{\partial\xi^2}-\frac{y}{(1+x)^3}\frac{\partial^2 u}{\partial\xi\partial\eta}+$$

$$+\frac{1}{(1+x)^2}\frac{\partial^2 u}{\partial\eta^2}-\frac{1}{2(1+x)^2}\frac{\partial u}{\partial\xi},$$

$$\frac{\partial^2 u}{\partial x\partial y}=-\frac{y(1+x+y^2)}{4(1+x)^5}\frac{\partial^2 u}{\partial\xi^2}+\frac{1+x+2y^2}{2(1+x)^4}\frac{\partial^2 u}{\partial\xi\partial\eta}-$$

$$-\frac{y}{(1+x)^3}\frac{\partial^2 u}{\partial \eta^2}+\frac{y}{(1+x)^3}\frac{\partial u}{\partial \xi}-\frac{1}{(1+x)^2}\frac{\partial u}{\partial \eta},$$

tengliklarga ega bo'lamiz. Bu ifodalarni berilgan tenglamaga olib borib qo'ysak, u holda berilgan tenglamaning

$$\xi\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}+\frac{1}{2}\frac{\partial u}{\partial \xi}=0$$

kanonik ko'rinishini hosil qilamiz.

12. Ikki o'zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglamalarning umumiy yechimini topishga doir misollar. Bu yerda ikki o'zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglamalarning umumiy yechimini topishga doir misollarni qaraymiz.

1–misol. $u_{xy}=0$ tenglamaning umumiy yechimini toping.

Yechish: $u_x=v$ deb belgilash kiritamiz. U holda $v_y=0$ tenglamaga ega bo'lamiz. Ushbu tenglamani yechish uchun uni integrallaymiz va $u_x=C(x)$ tenglikka ega bo'lamiz. Topilgan ifodani kiritilgan belgilashga olib, borib qo'yib, $u_x=C(x)$ tenglamaga ega bo'lamiz. Bu tenglamani integrallab $u(x,y)=f(x)+g(y)$ umumiy yechimga ega bo'lamiz, bu yerda $f(x)$ va $g(x)$ funksiyalar ixtiyoriy differensiallanuvchi funksiyalardir.

2–misol. $u_{xx}-2u_{xy}-3u_{yy}=0$ tenglamaning umumiy yechimini toping.

Yechish: $A=1, B=-1, C=-3,\quad \Delta=B^2-AC=1-1\cdot(-3)=4>0$ bo'lgani uchun yuqoridagi tenglama giperbolik tipdagi tenglama ekan. Endi esa xarakteristik tenglamasini tuzamiz va uni yechamiz:

$$y'^2+2y'-3=0,\quad y'=\frac{-2\pm\sqrt{4+12}}{2}=-1\pm 2,\quad y'=-3,\quad y'=1$$

$$y=-3x-C_1,\quad y=x+C_2,\quad C_1=3x+y,\quad C_2=x-y,$$

bu yerda C_1, C_2 o'zgarmaslarni mos ravishda ξ *va* η lar bilan almashtiramiz, ya'ni

$$\begin{cases} \xi = 3x + y \\ \eta = x - y \, . \end{cases}$$

U holda u funksiyani murakkab funksiya deb qarab ξ *va* η o'zgaruvchilar x *va* y o'zgaruvchilarning chiziqli funksiyalari ekanligini hisobga olib birinchi va ikkinchi tartibli xususiy hosilalarni hisoblaymiz:

$$u_x = 3u_\xi + u_\eta, \quad u_y = u_\xi - u_\eta, \quad u_{xx} = 9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta},$$

$$u_{xy} = 3u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta}, \quad u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}.$$

Topilgan ifodalarni $u_{xx} - 2u_{xy} - 3u_{yy} = 0$ tenglamaga olib borib qo'yamiz. Natijada

$$9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta} - 2\cdot\left(3u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta}\right) - 3\cdot\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right) = 0,$$

yoki $16u_{\xi\eta} = 0$, yoki $u_{\xi\eta} = 0$ tenglamaga ega bo'lamiz. Ushbu tenglamaning umumiy yechimi yuqoridagi 1-misolga asosan quyidagicha bo'ladi:

$$u(\xi, \eta) = f(\xi) + g(\eta).$$

Bu yerda ξ *va* η o'zgaruvchilar o'rniga ularning x *va* y o'zgaruvchilar orqali ifodalarini qo'yib,

$$u(x, y) = f(3x + y) + g(x - y)$$

umumiy yechimga ega bo'lamiz.

3-misol. $u_{xy} - 2u_x = 0$ tenglamaning umumiy yechimini toping.

Yechish: $u_x = v$ deb belgilash kiritamiz. U holda $v_y - 2v = 0$ tenglamaga ega bo'lamiz. Ushbu chiziqli tenglamani yechamiz va $v = C(x)\,e^{2y}$ tenglikka ega bo'lamiz. Topilgan ifodani kiritilgan belgilashga olib borib qo'yib, $u_x = C(x)e^{2y}$ tenglamaga ega bo'lamiz. Bu chiziqli bir jinsli tenglamani yechsak, $u(x, y) = f(x)e^{2y} + g(y)$ umumiy yechimni hosil qilamiz, bu yerda $f(x)$ va $g(x)$ funksiyalar ixtiyoriy differensiallanuvchi funksiyalardir.

4-misol. $u_{xy} - 2u_x - 3u_y + 6u = 2e^{x+y}$ tenglamaning umumiy yechimini toping.

Yechish: Ushbu tenglamani yechish uchun avval tenglamadagi birinchi tartibli xususiy hosilalarni yo'qotamiz. Buning uchun $u(x,y)=v(x,y)\cdot e^{\lambda x+\mu y}$ almashtirish bajaramiz, bu yerdagi λ va μ o'zgarmaslarni keyinchalik tanlaymiz. Birinchi va ikkinchi tartibli xususiy hosilalarni hisoblaymiz:

$$u_x = v_x \cdot e^{\lambda x+\mu y} + v\cdot\lambda e^{\lambda x+\mu y}, \quad u_y = v_y\cdot e^{\lambda x+\mu y} + v\cdot\mu e^{\lambda x+\mu y},$$

$$u_{xy} = v_{xy}\cdot e^{\lambda x+\mu y} + v_x\cdot\mu e^{\lambda x+\mu y} + v_y\cdot\lambda e^{\lambda x+\mu y} + v\cdot\lambda\mu e^{\lambda x+\mu y}$$

Topilgan ifodalarni $u_{xy}-2u_x-3u_y+6u=2e^{x+y}$ tenglamaga olib borib qo'yamiz. Natijada

$$v_{xy}\cdot e^{\lambda x+\mu y}+v_x\cdot\mu e^{\lambda x+\mu y}+v_y\cdot\lambda e^{\lambda x+\mu y}+v\cdot\lambda\mu e^{\lambda x+\mu y}-$$
$$-2\cdot\left(v_x\cdot e^{\lambda x+\mu y}+v\cdot\lambda e^{\lambda x+\mu y}\right)-3\cdot\left(v_y\cdot e^{\lambda x+\mu y}+v\cdot\mu e^{\lambda x+\mu y}\right)+$$
$$+6v\cdot e^{\lambda x+\mu y}=2e^{x+y},$$

yoki

$$v_{xy}\cdot e^{\lambda x+\mu y}+(\mu-2)v_x\cdot e^{\lambda x+\mu y}+(\lambda-3)v_y\cdot e^{\lambda x+\mu y}+$$
$$+(\lambda\mu-2\lambda-3\mu+6)v\cdot e^{\lambda x+\mu y}=2e^{x+y}$$

tenglamaga ega bo'lamiz. Ushbu tenglikda λ va μ o'zgarmaslarni shunday tanlaymizki, oxirgi tenglikda birinchi tartibli xususiy hosilalar qatnashmasin. Buning uchun $\lambda=3,\ \mu=2$ deb tanlaymiz va quyidagi tenglamaga ega bo'lamiz:

$$e^{3x+2y}\cdot v_{xy}=2e^{x+y},$$

ya'ni $v_{xy}=2e^{-2x-y}$. Bu yerda $v_x=\varphi$ deb belgilash kiritamiz. U holda ushbu

$$\varphi_y=2e^{-2x-y}$$

chiziqli tenglamaga ega bo'lamiz va uni yechamiz. Natijada

$$\varphi=-2e^{-2x-y}+C(x),$$

ya'ni $v_x=-2e^{-2x-y}+C(x)$ chiziqli tenglamaga ega bo'lamiz. Bu tenglamani integrallab, $v=e^{-2x-y}+f(x)+g(y)$ ekanligini hosil qilamiz.

$$u(x,y) = v(x,y) \cdot e^{3x+2y}$$

almashtirishga asosan,

$$u(x,y) = e^{x+y} + \left[f(x) + g(y)\right] \cdot e^{3x+2y}$$

umumiy yechimni hosil qilamiz, bu yerda $f(x)$ va $g(x)$ funksiyalar ixtiyoriy differensiallanuvchi funksiyalardir.

13. Laplasning kaskad usuli. Quyidagi

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y) \tag{53}$$

tenglamani qaraylik, bu yerda a, b, c koeffitsientlar va f oldindan berilgan, hamda x va y o'zgaruvchilarga bog'liq funksiyalardir.

Agar bu a, b, c koeffitsientlar uchun

$$h \equiv \frac{\partial a}{\partial x} + ab - c \equiv 0 \tag{54}$$

ayniyat o'rinli bo'lsa, (53) tenglamani quyidagi ko'rinishda yozish mumkin:

$$\frac{\partial v}{\partial x} + bv = f, \tag{55}$$

bu yerda

$$v = \frac{\partial u}{\partial y} + au \tag{56}$$

bo'ladi. Bundan, esa berilgan xususiy hosilali differensial tenglamaning umumiy yechimini

$$u = e^{-\int ady}\left\{X + \int\left\{Y + \int f \cdot e^{\int bdx}dx\right\}e^{\int ady - \int bdx}dy\right\} \tag{57}$$

shaklda hosil qilamiz, bu yerda X va Y – ixtiyoriy funksiyalar bo'lib, mos ravishda x va y ga bog'liq. Xuddi shunga o'xshash, agar

$$k \equiv \frac{\partial b}{\partial y} + ab - c \equiv 0 \tag{58}$$

ayniyat o'rinli bo'lsa, (53) tenglamani quyidagi ko'rinishda yozish mumkin:

$$\frac{\partial v}{\partial y} + av = f, \tag{59}$$

bu yerda

$$v=\frac{\partial u}{\partial x}+bu \qquad (60)$$

bo'ladi. Bundan, esa berilgan xususiy hosilali differensial tenglamaning umumiy yechimini

$$u=e^{-\int bdx}\left\{Y+\int\left\{X+\int f\cdot e^{\int ady}dy\right\}e^{\int bdx-\int ady}dx\right\} \qquad (61)$$

shaklda hosil qilamiz, bu yerda X va Y – ixtiyoriy funksiyalar bo'lib, mos ravishda x va y ga bog'liq.

$h\neq 0$ bo'lgan holda (53) tenglamaga o'xshash quyidagi tenglama qaraladi:

$$L_1u_1=\frac{\partial^2 u_1}{\partial x\partial y}+a_1\frac{\partial u_1}{\partial x}+b_1\frac{\partial u_1}{\partial y}+c_1u_1=f_1, \qquad (62)$$

bu yerda

$$a_1=a-\frac{\partial \ln h}{\partial y},\quad b_1=b,$$

$$c_1=c-\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}-b\frac{\partial \ln h}{\partial y},\quad f_1=f\cdot\left(a-\frac{\partial \ln h}{\partial y}\right). \qquad (63)$$

Agar u_1 funksiyani topish mumkin bo'lsa, u holda qaralayotgan (53) tenglamaning yechimi quyidagi formula bilan topiladi:

$$u=\frac{\frac{\partial u_1}{\partial x}+bu_1-f}{h}. \qquad (64)$$

(62) tenglama uchun

$$h_1=2h-k-\frac{\partial^2 \ln h}{\partial x\partial y},\quad k_1=h \qquad (65)$$

bo'ladi. Agar $h_1=0$ bo'lsa, u holda u_1 funksiya yuqorida ifodalangan usul bo'yicha hosil qilinadi. Agar $h_1\neq 0$ bo'lsa, u holda yuqoridagi jarayonni davom ettiramiz va yuqoridagiga o'xshash $L_2u_2=f_2$ tenglamani hosil qilamiz va hokazo.

$k\neq 0$ bo'lgan holda yuqoridagiga o'xshash quyidagi tenglamalar zanjirini hosil qilish mumkin: $L_{-1}u_{-1}=f_{-1}$, $L_{-2}u_{-2}=f_{-2}$ va hokazo.

Agar qandaydir h_i (yoki k_i) lar $i-$ qadamda nolga aylansa, (53) tenglamaning umumiy yechimini topish mumkin bo'ladi.

1-misol. Yuqoridagi usul yordamida

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\beta'}{x-y}\frac{\partial u}{\partial x} + \frac{\beta}{x-y}\frac{\partial u}{\partial y} = 0 \quad (66)$$

Eyler–Darbu tenglamasini β yoki β' koeffitsientlardan birortasi butun son bo'lsa, u holda bu tenglamani yechamiz, bu yerda $a(x,y) = -\frac{\beta'}{x-y}$, $b(x,y) = \frac{\beta}{x-y}$, $c(x,y) = 0$, $f(x,y) = 0$. Endi (54) formula bilan aniqlangan h ni hisoblaymiz va quyidagi tenglikni hosil qilamiz:

$$h = \frac{\beta'(1-\beta)}{(x-y)^2}.$$

Agar $\beta = 1$ bo'lsa, $h = 0$ bo'ladi. Agar $h \neq 0$ bo'lsa, (63) ga ko'ra quyidagilarni topamiz:

$$a_1 = -\frac{2+\beta'}{x-y},\quad b_1 = \frac{\beta}{x-y},\quad c_1 = -\frac{\beta'+\beta}{(x-y)^2},\qquad f_1 = 0.$$

Bundan $h_1 = \frac{(1+\beta')(2-\beta)}{(x-y)^2}$ bo'ladi. Agar $\beta = 2$ bo'lsa, u holda $h_1 = 0$ bo'ladi va hokazo. Umuman olganda,

$$E(\alpha,\beta) \equiv \frac{\partial^2 u}{\partial x \partial y} - \frac{\beta}{x-y}\frac{\partial u}{\partial x} + \frac{\alpha}{x-y}\frac{\partial u}{\partial y} = 0 \quad (67)$$

Eyler–Darbu tenglamasining umumiy yechimini ixtiyoriy α va β haqiqiy sonlar uchun ham hosil qilish mumkin. Biz bu yerda α va β natural sonlar, yoki musbat bo'lmagan butun sonlar bo'lgan holda umumiy yechimni topamiz. Shu maqsadda (67) tenglamani

$$(x-y)\frac{\partial^2 u}{\partial x \partial y} - \beta\frac{\partial u}{\partial x} + \alpha\frac{\partial u}{\partial y} = 0 \quad (68)$$

shaklda yozamiz. (68) tenglamani x bo'yicha differensiallab,